Due: Thursday, March 26 at the beginning of class.

Use Mathematica as you wish for computations and for plotting. Provide complete documentation by appending your Mathematica notebook. Develop the solutions in a separate and complete exposition, referring the reader to the Mathematica work, transcribing your results and stating as needed that they were obtained using the software. DO NOT simply use Mathematica and expect the reader/ grader to dig out your answers from your Mathematica output. The Mathematica notebook should serve as an appendix documenting your computations and exhibiting your plots.

As a first step, state the general separation of variables series solution for the type of problem being considered (i.e. zero temperature ends or insulated ends).

1. (a) Solve the following initial-boundary value problem.

\[
\frac{\partial u(x,t)}{\partial t} = 0.1 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 3, \quad 0 < t < \infty
\]

\[
u(0,t) = u(3,t) = 0, \quad 0 \leq t < \infty
\]

\[
u(x,0) = \begin{cases} 
0, & 0 \leq x < 1 \\
20(x-1), & 1 \leq x \leq \frac{x}{3} \\
20(2-x), & \frac{x}{3} < x \leq 2 \\
0, & 2 < x \leq 3
\end{cases}
\]

(b) Plot \(u(x,0), u(x,2)\) and \(u(x,8)\) on the same graph for \(0 \leq x \leq 3\). Use a 100-term partial sum to approximate the solution.

(c) At what time does \(u(x,3), t = 3\)? (Again use the 100-term partial sum approximation of the solution. The Mathematica FindRoot command can prove useful.)

2. (a) Solve the following initial-boundary value problem.

\[
\frac{\partial u(x,t)}{\partial t} = 0.1 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 3, \quad 0 < t < \infty
\]

\[
\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(3,t)}{\partial x} = 0, \quad 0 \leq t < \infty
\]

\[
u(x,0) = \begin{cases} 
0, & 0 \leq x < 1 \\
20(x-1), & 1 \leq x \leq \frac{x}{3} \\
20(2-x), & \frac{x}{3} < x \leq 2 \\
0, & 2 < x \leq 3
\end{cases}
\]

(b) Plot \(u(x,0), u(x,2)\) and \(u(x,8)\) on the same graph for \(0 \leq x \leq 3\). Use a 100-term series partial sum to approximate the solution.

(c) Evaluate \(\int_0^1 u(x,2) dx\), a measure of the energy content in the left third of the domain at time 2.

(Again use the 100-term series partial sum approximation of the solution.)
3. Discussion:

We consider the problem of a solid homogeneous sphere, having radius $r_0$, initially at temperature $T_0$, suddenly immersed into a constant zero temperature bath. The goal is to determine how heat diffuses out of the sphere and, in particular, how the temperature at or near the center of the sphere decreases as a function of time.

Because of the problem symmetries the temperature within the sphere will be radially symmetric, depending only upon the radial coordinate and time.

Problem Formulation:

Let $u(r,t)$ denote the temperature within the sphere at radial location $r$ and time $t$. The radially symmetric heat equation in spherical coordinates is:

$$
\frac{\partial u(r,t)}{\partial t} = \kappa \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u(r,t)}{\partial r} \right), \quad 0 < r < r_0, \quad 0 < t < \infty
$$

The temperature at the surface of the sphere is kept at the zero bath temperature. We therefore have the boundary condition

$$
u(r_0,t) = 0, \quad 0 \leq t < \infty.
$$

We also require $u(r,t)$ to remain bounded within the sphere. As an initial condition, we assume that by the time our clock starts at $t = 0$ the temperature within the sphere has become:

$$
u(r,0) = T_0 \left( 1 - e^{-\frac{r}{\delta}} \right), \quad 0 \leq r \leq r_0
$$

where $0 < \delta \ll r_0$. Thus $\delta$ represents an initial thermal skin depth.

For definiteness assume that:

$$
\begin{align*}
    r_0 &= 2, \quad T_0 = 100, \quad \kappa = 0.1, \quad \delta = 0.05
\end{align*}
$$

(a) Introduce the change of dependent variable $u(r,t) = \frac{U(r,t)}{r}$. Show that $U(r,t)$ is a solution of:

$$
\frac{\partial U(r,t)}{\partial t} = 0.1 \frac{\partial^2 U(r,t)}{\partial r^2}, \quad 0 < r < 2, \quad 0 < t < \infty
$$

The dependent variable $U(r,t)$ satisfies zero temperature boundary conditions at $r = 0$ and $r = 2$. Since $u(r,t)$ must be bounded within the sphere and $U(r,t) = ru(r,t)$ we must have:

$$
U(0,t) = 0 \quad \text{and} \quad U(2,t) = 2u(2,t) = 0, \quad 0 \leq t < \infty.
$$

The initial condition $U(r,0)$ becomes:
\[ U(r,0) = ru(r,0) = rT_0 \left( 1 - e^{-\frac{r-r_0}{\delta}} \right) = 100r \left( 1 - e^{-2(2-r)} \right), \quad 0 \leq r \leq 2. \]

**Problem:**

The problem for \( U(r,t) \) becomes the following **Zero Temperature Ends problem**:

\[
\frac{\partial U(r,t)}{\partial t} = 0.1 \frac{\partial^2 U(r,t)}{\partial r^2}, \quad 0 < r < 2, \quad 0 < t < \infty
\]

\[ U(0,t) = 0, \quad U(2,t) = 0, \quad 0 \leq t < \infty \]

\[ U(r,0) = 100r \left( 1 - e^{-2(2-r)} \right), \quad 0 \leq r \leq 2 \]

**Solve this problem as follows:**

(b) Formulate and solve the zero temperature ends problem for \( U(r,t) \).

The desired solution \( u(r,t) = \frac{U(r,t)}{r} \).

Note that the solution \( u(r,t) = \frac{U(r,t)}{r} \) that you obtain will have the form

\[
u(r,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{2}\right)^2 t} \sin\left(\frac{n\pi r}{2}\right)
\]

which is indeterminate at \( r = 0 \). To avoid this indeterminacy, we will study the decrease in temperature at \( r = 0.01 \) rather than \( r = 0 \) and plot the temperature profile at various times over the radial interval \( 0.01 \leq r \leq 2 \).

**Using the first 500 terms to approximate the infinite series:**

(c) Plot \( u(0.01,t) \) for \( 0 \leq t \leq 20 \).

(d) Plot \( u(r,0), u(r,2) \) and \( u(r,10) \) on the same graph over the interval \( 0.01 \leq r \leq 2 \).
1. \( u(x, t) = 0.1 u_x(x, t), 0 < x < 3, 0 < t < \infty \)
\( u(0, t) = 0, u(3, t) = 0, 0 \leq t < \infty \)
\( u(x, 0) = f(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ 20(x-1), & 1 \leq x \leq 2 \\ 20(2-x), & 2 \leq x \leq 3 \\ 0, & 2 < x \leq 3 \end{cases} \)

a) \( u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{3}\right)^2 0.1t} \sin\left(\frac{n\pi x}{3}\right) \)

\[ c_n = \frac{2}{3} \int_0^1 f(x) \sin\left(\frac{n\pi x}{3}\right) dx = \frac{2}{3} \left[ \int_0^{\frac{3}{4}} 20(x-1) \sin\left(\frac{n\pi x}{3}\right) dx + \int_{\frac{3}{4}}^{\frac{2}{3}} 20(2-x) \sin\left(\frac{n\pi x}{3}\right) dx \right] \]

\[ = -120 \left[ \frac{\sin\left(\frac{n\pi}{4}\right)}{\frac{n\pi}{4}} - 2 \sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{2n\pi}{3}\right) \right] \]

(b) See Mathematica for the plot.

(c) Find \( t \) such that \( \sum_{n=1}^{100} c_n e^{-\left(\frac{n\pi}{3}\right)^2 0.1t} \sin\left(\frac{n\pi x}{2}\right) = 3 \)

Using FindRoot (Mathematica): \( t \approx 1.999 \)

2. \( u(x, t) = 0.1 u_x(x, t), 0 < x < 3, 0 < t < \infty \)
\( u(0, t) = u_x(3, t) = 0, 0 \leq t < \infty \)

Insulated ends
\( k = 0.1, \lambda = 3 \)

\( u(x, 0) = f(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ 20(x-1), & 1 \leq x \leq \frac{3}{2} \\ 20(2-x), & \frac{3}{2} \leq x \leq 2 \\ 0, & 2 < x \leq 3 \end{cases} \)

a) \( u(x, t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{3}\right)^2 0.1t} \cos\left(\frac{n\pi x}{3}\right) \)

\[ c_n = \frac{2}{3} \int_0^3 f(x) \cos\left(\frac{n\pi x}{3}\right) dx = \frac{2}{3} \left[ \int_0^{\frac{3}{4}} 20(x-1) \cos\left(\frac{n\pi x}{3}\right) dx + \int_{\frac{3}{4}}^{\frac{2}{3}} 20(2-x) \cos\left(\frac{n\pi x}{3}\right) dx \right] \]

\[ = \frac{480 \cos\left(\frac{n\pi}{2}\right) \sin^2\left(\frac{n\pi}{12}\right)}{n^2 \pi^2} \]

with: \( c_0 = \lim_{n \to 0} \left[ \frac{480 \cos\left(\frac{n\pi}{2}\right) \sin^2\left(\frac{n\pi}{12}\right)}{n^2 \pi^2} \right] = 10/3 \)

(b) See Mathematica for the plot.

c) \( \int_0^1 u(x, 2) dx \approx c_0 + \frac{1}{2} \int_0^1 f(x) dx + \sum_{n=1}^{100} c_n e^{-\left(\frac{n\pi}{3}\right)^2 0.1(2)} \int_0^{1/2} \cos\left(\frac{n\pi x}{3}\right) dx = 1.13 \)
3. a) \( \frac{\partial u}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) \). Let \( u(r,t) = \frac{U(r,t)}{r} \). Then:

\[
\frac{\partial U}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) = -\frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) = -\frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r} \frac{r^2 \frac{\partial^2 U}{\partial r^2}}{r^2} = -\frac{1}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial r^2};
\]

\[
\frac{r^2}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) = r^2 \frac{\partial^2 U}{\partial r^2};
\]

\[
\frac{\partial U}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) \Rightarrow \frac{1}{r} \frac{\partial U}{\partial t} = \frac{k}{r} \frac{\partial^2 U}{\partial r^2} \Rightarrow \frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial r^2}.
\]

b) \( \frac{\partial U}{\partial t} = 0, 1 \frac{\partial^2 U}{\partial r^2}, 0 < r < 2, 0 < t < \infty \)

\( U(0,t) = U(2,t) = 0, 0 < t < \infty \)

\( U(r,0) = 100r \left( 1 - e^{-20(2-r)} \right), 0 \leq r \leq 2 \)

\[
U(r,t) = \sum_{n=1}^{\infty} c_n e^{-(n \pi^2 / 2) \frac{r^2}{4}} \sin \left( \frac{n \pi r}{2} \right)
\]

\[
c_n = \frac{2}{\pi} \int_0^2 100r \left( 1 - e^{-20(2-r)} \right) \sin \left( \frac{n \pi r}{2} \right) dr \quad \text{(See Mathematica)}
\]

\( U(r,t) = \sum_{n=1}^{\infty} c_n e^{-(n \pi^2 / 2) \frac{r^2}{4}} \frac{\sin \left( \frac{n \pi r}{2} \right)}{r} \)

c) (e, d): See Mathematica for plots.
Problem 1:

(a):

\[
\frac{2}{3} \left( -\frac{30 \left( n \pi \cos \left( \frac{n \pi}{2} \right) + 6 \sin \left( \frac{n \pi}{3} \right) - 6 \sin \left( \frac{n \pi}{2} \right) \right)}{n^2 \pi^2} + \frac{30 \left( n \pi \cos \left( \frac{n \pi}{2} \right) + 6 \sin \left( \frac{n \pi}{2} \right) - 6 \sin \left( \frac{2n \pi}{3} \right) \right)}{n^2 \pi^2} \right)
\]

Simplify \( \frac{2}{3} \)

\[
-\frac{30 \left( n \pi \cos \left( \frac{n \pi}{2} \right) + 6 \sin \left( \frac{n \pi}{3} \right) - 6 \sin \left( \frac{n \pi}{2} \right) \right)}{n^2 \pi^2} + \frac{30 \left( n \pi \cos \left( \frac{n \pi}{2} \right) + 6 \sin \left( \frac{n \pi}{2} \right) - 6 \sin \left( \frac{2n \pi}{3} \right) \right)}{n^2 \pi^2}
\]

\[
120 \left( \sin \left( \frac{n \pi}{3} \right) - 2 \sin \left( \frac{n \pi}{2} \right) + \sin \left( \frac{2n \pi}{3} \right) \right)
\]

\[
cl[n_] := -\frac{120 \left( \sin \left( \frac{n \pi}{3} \right) - 2 \sin \left( \frac{n \pi}{2} \right) + \sin \left( \frac{2n \pi}{3} \right) \right)}{n^2 \pi^2};
\]

\[ ul[x_, t_, N_] :=
\;\; Sum[cl[n] \times \text{Exp}[-(n \times \frac{\pi}{3})^2 \times 0.1 \times t] \times \sin[n \times \pi \times x / 3], \{n, 1, N\}];
\]

(b):

\[ \text{Plot}[[ul[x, 0, 100], ul[x, 2, 100], ul[x, 8, 100]], \{x, 0, 3\}, \text{PlotRange} \rightarrow \{0, 10\}] \]

(c):
Plot[u1[3/2, t, 100], {t, 0, 20}, PlotRange -> {0, 10}]

FindRoot[u1[3/2, t, 100] == 3, {t, 4}]

{t → 1.99876}

Problem 2:
(a):

\[
\frac{2}{3} \left( \frac{30 \left( -6 \cos \left( \frac{n \pi}{3} \right) + 6 \cos \left( \frac{n \pi}{2} \right) + n \pi \sin \left( \frac{n \pi}{2} \right) \right)}{n^2 \pi^2} - \frac{30 \left( -6 \cos \left( \frac{2n \pi}{3} \right) + 6 \cos \left( \frac{2n \pi}{2} \right) + n \pi \sin \left( \frac{2n \pi}{2} \right) \right)}{n^2 \pi^2} \right)
\]

\[
\text{Simplify} \left[ \frac{2}{3} \left( \frac{30 \left( -6 \cos \left( \frac{n \pi}{3} \right) + 6 \cos \left( \frac{n \pi}{2} \right) + n \pi \sin \left( \frac{n \pi}{2} \right) \right)}{n^2 \pi^2} - \frac{30 \left( -6 \cos \left( \frac{2n \pi}{3} \right) + 6 \cos \left( \frac{2n \pi}{2} \right) + n \pi \sin \left( \frac{2n \pi}{2} \right) \right)}{n^2 \pi^2} \right) \right]
\]

\[
\frac{480 \cos \left( \frac{n \pi}{3} \right) \sin \left( \frac{n \pi}{12} \right)^2}{n^2 \pi^2}
\]

c2[n_] := \frac{480 \cos \left( \frac{n \pi}{2} \right) \sin \left( \frac{n \pi}{12} \right)^2}{n^2 \pi^2} ;

c2[0] = Limit[c2[n], n → 0]

\[
\frac{10}{3}
\]

\[u_2[x_, t_, N_] := \text{c2[0]} / 2 + \text{Sum}[\text{c2}[n] \ast \text{Exp}[-((n \ast \text{Pi} / 3) \ast t)] \ast \text{Cos}[n \ast \text{Pi} \ast x / 3], \{n, 1, N\}]\]
(b):

\[
\text{Plot[\{u2[x, 0, 100], u2[x, 2, 100], u2[x, 8, 100]\}, \{x, 0, 3\}, \text{PlotRange \to \{0, 10\}}]
\]

(c):

\[
\text{Integrate[u2[x, 2, 100], \{x, 0, 1\}}]
\]

1.12982

Problem 3:

\[
f[r_] := 100 \times (1 - r \times \text{Exp[-20 \times (2 - r)]});
\]

\[
\text{Plot[f[r], \{r, 0, 2\}, \text{PlotRange \to \{0, 100\}}]
\]
\[\text{Integrate}[r \cdot f[r] \cdot \text{Sin}[n \cdot \text{Pi} \cdot r / 2], \{r, 0, 2\}] \]

\[= \frac{400 \left(n \cdot \pi \cdot \text{Cos}[n \cdot \pi] - \text{Sin}[n \cdot \pi]\right)}{n^2 \pi^2} + \]
\[\frac{1}{e^{40} \left(1600 + n^2 \pi^2\right)^3} \cdot 800 \left(e^{40} \cdot n \cdot \pi \left(2313600 + 3038 \cdot n^2 \pi^2 + n^4 \pi^4\right) \cdot \text{Cos}[n \cdot \pi] - 2 \left(4800 \cdot n \cdot \pi - n^3 \pi^3 + e^{40} \left(48704000 + 63880 \cdot n^2 \pi^2 + 21 \cdot n^4 \pi^4\right) \cdot \text{Sin}[n \cdot \pi]\right)\]

Since \(n\) is an integer, \(\text{Sin}(n\pi) = 0\) and \(\text{Cos}(n\pi) = (-1)^n\):

\[c3[n_] := -\frac{400 \left(n \cdot \pi \cdot (-1)^n\right)}{n^2 \pi^2} + \]
\[\frac{1}{e^{40} \left(1600 + n^2 \pi^2\right)^3} \cdot 800 \left(e^{40} \cdot n \cdot \pi \left(2313600 + 3038 \cdot n^2 \pi^2 + n^4 \pi^4\right) \cdot (-1)^n\right);\]

\[U[r_, t_, N_] := \text{Sum}[c3[n] \cdot \text{Exp}[-((n \cdot \text{Pi} / 2)^2) \cdot 0.1 \cdot t] \cdot \text{Sin}[n \cdot \text{Pi} \cdot r / 2], \{n, 1, N\}];\]
\[u3[r_, t_, N_] := U[r, t, N] / r;\]

Check to see how the solution partial sum replicates the initial condition.

\[\text{Plot}[u3[r, 0, 500], \{r, 0.05, 2\}, \text{PlotRange} \to \{-0.1, 100\}]\]

(d):
(e):

```math
Plot[{u3[x, 0, 500], u3[x, 2, 500], u3[x, 10, 500]},
     {x, 0.01, 2}, PlotRange -> (-0.1, 100)]
```