Part I (Takehome): to be submitted Tuesday, April 29, at the beginning of class.  
(Part II, the in-class portion, will be taken during the 4/29 class.)

Answer all questions. Show all work. No credit will be given for unsupported answers. You may consult your text and/or class notes, but you must work alone. Do not discuss any aspect of this test with anyone but the instructor. Use Mathematica as you wish for computations and for plotting as directed. Provide complete documentation by appending your Mathematica notebook to your submitted work. Label the problem part being solved. 
Place your answers on separate sheets. Use this question sheet as a cover sheet.  
Staple your work together in the upper left corner.

For each problem, state the problem being solved and the appropriate separation of variables solution series before you start evaluating coefficients.

1. (a) Solve the initial-boundary value problem:

\[
\frac{\partial u(x,t)}{\partial t} = 0.1 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 3, \quad 0 < t < \infty
\]
\[u(0,t)=u(3,t)=0, \quad 0 \leq t < \infty\]
\[u(x,0)=f(x)=\begin{cases} \frac{x}{3}, & 0 \leq x \leq 2 \\ -x+3, & 2 < x \leq 3 \end{cases}\]

(b) Use Mathematica to plot the solution at times \( t = 0 \) and \( t = 2 \) on the same graph, using the sum of the first 100 terms as an approximation of the true solution.

(15 points)

2. (a) Solve the initial-boundary value problem:

\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{4} \frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < 3, \quad 0 < t < \infty
\]
\[u(0,t)=u(3,t)=0, \quad 0 \leq t < \infty\]
\[u(x,0)=f(x)=\begin{cases} \frac{x}{3}, & 0 \leq x \leq 2 \\ -x+3, & 2 < x \leq 3 \end{cases}, \quad \frac{\partial u(x,0)}{\partial t} = \sin\left(\frac{2\pi x}{3}\right), \quad 0 \leq x \leq 3\]

(b) Use Mathematica to plot the solution at times \( t = 0 \) and \( t = 1 \) on the same graph, using the sum of the first 100 terms as an approximation of the true solution.

(20 points)

Problem 3 is on the back of this page.
3. Consider the following problem for Laplace's equation with Dirichlet boundary conditions.

\[ \nabla^2 u(x,y) = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1 \]

\[ u(1,y) = \cos\left(\frac{\pi y}{2}\right), \quad 0 \leq y \leq 1 \]

with boundary conditions:

\[ u(x,1) = 0, \quad 0 \leq x \leq 1 \]

\[ u(0,y) = 0, \quad 0 \leq y \leq 1 \]

\[ u(x,0) = x, \quad 0 \leq x \leq 1 \]

(a) Determine a solution of Laplace's equation, \( v(x,y) \), that takes on the boundary vertex values.

(b) Form \( U(x,y) = u(x,y) - v(x,y) \), formulate and solve the resulting boundary value problem for \( U(x,y) \) and form the desired solution \( u(x,y) \).

(c) Use Mathematica's Plot3D command to plot the solution surface \( u(x,y) \), using the sum of the first 100 terms as an approximation of the true solution.

(20 points)
1. (a) $u_t = \Delta u$, $0 < x < 3, 0 < t < \infty$  
 heat equation, $\lambda = 3, \kappa = 0.1$  
 $u(0,t) = u(3,t) = 0, 0 \leq t < \infty$ zero temperature ends  
 
 $u(x,0) = f(x) = \begin{cases} 
 \frac{x}{2}, & 0 \leq x \leq 2 \\
 -x + 3, & 2 < x \leq 3 
\end{cases}$  
 
 $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(\frac{n\pi}{3})^2 \kappa t} \sin \left( \frac{n\pi}{3}x \right)$  
 
 $u(x,0) = \frac{2}{3} f(x) \sin \left( \frac{n\pi}{3}x \right) dx = \frac{2}{3} \left( \int_0^2 \frac{x}{2} \sin \left( \frac{n\pi}{3}x \right) dx + \int_2^3 (-x+3) \sin \left( \frac{n\pi}{3}x \right) dx \right)$  
 
 $= \frac{9 \sin \left( \frac{2n\pi}{3} \right)}{n^2 \pi^2}$ (Mathematica)  
 
 $\therefore \quad u(x,t) = \sum_{n=1}^{\infty} \frac{9 \sin \left( \frac{2n\pi}{3} \right)}{n^2 \pi^2} e^{-(\frac{n\pi}{3})^2 \kappa t} \sin \left( \frac{n\pi}{3}x \right)$  
 
 b) See Mathematica for plot.

2. (a) $u_t = \frac{1}{4} u_{xx}, \quad 0 < x < 3, \quad 0 < t < \infty$ wave equation $c = 2, k = 3$  
 $u(0,t) = u(3,t) = 0, \quad 0 < t < \infty$ pinned string  
 $u(x,0) = f(x) = \begin{cases} 
 \frac{x}{2}, & 0 \leq x \leq 2 \\
 -x + 3, & 2 < x \leq 3 
\end{cases}$, $\quad u_t(x,0) = \sin \left( \frac{2n\pi}{3}x \right), \quad 0 \leq x \leq 3$  
 
 $u(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{mn\pi}{3}t \right) + B_n \sin \left( \frac{mn\pi}{3}t \right) \right) \sin \left( \frac{mn\pi}{3}x \right)$  
 
 $u(x,0) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{mn\pi}{3}x \right) = f(x) \quad \therefore \quad A_n = \frac{9 \sin \left( \frac{2n\pi}{3} \right)}{n^2 \pi^2}$ (problem 1)  
 
 $u_t(x,0) = \sum_{n=1}^{\infty} 2n\pi \frac{B_n}{3} \sin \left( \frac{mn\pi}{3}x \right) = \sin \left( \frac{2n\pi}{3} \right)$  
 
 $\therefore \quad 4\frac{B_2}{3} = 1, \quad B_n = 0, \quad n \neq 2.$  
 
 $u(x,t) = \sum_{n=1}^{\infty} \frac{9 \sin \left( \frac{2n\pi}{3} \right) \cos \left( \frac{2n\pi}{3}t \right) \sin \left( \frac{mn\pi}{3}x \right) + 3 \sin \left( \frac{4\pi}{3} \right) \sin \left( \frac{2n\pi}{3} \right) \sin \left( \frac{mn\pi}{3}x \right)}{n^2 \pi^2} \sin \left( \frac{mn\pi}{3}x \right)$  
 
 b) See Mathematica for plot.

3. $u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 1$  

BCs: $u(1,y) = \cos \left( \frac{n\pi}{2} \right), \quad 0 \leq y \leq 1$  
 $u(x,1) = 0, \quad 0 \leq x \leq 1$  
 $u(0,y) = 0, \quad 0 \leq y \leq 1$  
 $u(x,0) = x, \quad 0 \leq x \leq 1$  

$\Delta u = 0$  

$u(1,y) = \cos \left( \frac{n\pi}{2} \right)$
3. (cont) (a) Let \( u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy \)

\[
\begin{align*}
   u(0,0) &= a_0 = 0; \\
   u(1,0) &= a_0 + a_1 = a_1 = 1; \\
   u(0,1) &= a_0 + a_2 = a_2 = 0 \\
   u(1,1) &= a_0 + a_1 + a_2 + a_3 = 1 + a_3 = 0 \Rightarrow a_3 = -1 \\
\end{align*}
\]

\( \therefore u(x,y) = x + xy \)

(b) \( \Delta u(x,y) = u(x+y) - u(x,y) \). Therefore,

\[
\begin{align*}
\Delta u &= 0, \quad 0 < x < 1, 0 < y < 1 \\
U(1,0) &= u(1,0) - u(0,0) = \cos(\frac{\pi y}{2}) - 1 + y \\
U(x,1) &= u(x,1) - u(x,0) = 0 - 0 = 0 \\
U(0,0) &= u(0,0) - u(0,0) = 0 - 0 = 0 \\
\Delta u(x,0) &= u(x,0) - u(x,0) = x - x = 0 \\
\end{align*}
\]

\[
U(x,y) = \sum_{n=1}^{\infty} a_n \sinh(n \pi x) \sin(n \pi y)
\]

\[
\begin{align*}
U(1,y) &= \sum_{n=1}^{\infty} a_n \sinh(n \pi) \sin(n \pi y) = \cos(\frac{\pi y}{2}) - 1 + y \\
\end{align*}
\]

\[
a_n \sinh(n \pi) = 2 \int_{0}^{1} \left( \cos(\frac{\pi y}{2}) - 1 + y \right) \sin(n \pi y) \, dy = \frac{1}{n \pi (4n^2 - 1)}
\]

\[
\therefore u(x,y) = x - xy + \sum_{n=1}^{\infty} 2 \frac{\sinh(n \pi x) \sin(n \pi y)}{\sinh(n \pi) \cdot n \pi (4n^2 - 1)}
\]

(b) See Mathematica for plot.
Problem 1:

(a):

\[
\frac{2}{3} \left( -6 n \pi \cos \left( \frac{2 n \pi}{3} \right) + 9 \sin \left( \frac{2 n \pi}{3} \right) \right) + \frac{3 \left( n \pi \cos \left( \frac{2 n \pi}{3} \right) + 3 \sin \left( \frac{2 n \pi}{3} \right) - 3 \sin(n \pi) \right)}{n^2 \pi^2}
\]

Simplify[%]

\[
\frac{9 \sin \left( \frac{2 n \pi}{3} \right) - 6 \sin(n \pi)}{n^2 \pi^2}
\]

\[
a[n_] := \frac{3 \left( 3 \sin \left( \frac{2 n \pi}{3} \right) \right)}{n^2 \pi^2} ;
\]

since \( \sin(n^2 \pi) = 0 \) for integer \( n \).

\[
u[x_, t_, N_] := \text{Sum}[a[n] * \text{Exp}\left[-(n \pi / 3)^2 \right] * 0.1 * t * \sin(n \pi * x / 3), \{n, 1, N\}] ;
\]

(b):

\[
\text{Plot}[\{u[x, 0, 100], u[x, 2, 100]\}, \{x, 0, 3\}]
\]

Problem 2:

\[
u2[x_, t_, N_] :=
\text{Sum}[\left( 9 \sin(2 n \pi / 3) / (n \pi)^2 \right) * \cos(2 n \pi * t / 3) * \sin(n \pi * x / 3),
\{n, 1, N\}] + \left( 3 / (4 \pi) \right) * \sin(4 \pi * t / 3) * \sin(2 \pi * x / 3) ;
\]
Plot[{u2[x, 0, 100], u2[x, 1, 100]}, {x, 0, 3}]

Problem 3:

(a):
\[
\text{Integrate}\left[(\cos(\pi y/2) - 1 + y) \sin(n \pi y), \{y, 0, 1\}\right]
\]
\[
n \pi + \left\{-1 - 2 n^2 \left(-2 + \pi^2\right)\sin(n \pi)\right\}
\]
\[
\frac{n^2 (-1 + 4 n^2) \pi^2}{n \pi + \left(-1 - 2 n^2 \left(-2 + \pi^2\right)\sin(n \pi)\right)}
\]
\[
a3[n_] := (2 / \text{Sinh}[n \pi \pi]) \times (1 / (n \pi \pi \pi (4 \pi n^2 - 1)))
\]
since \(\sin(n \pi) = 0\) for integer \(n\).

(b):
\[
u3[x_, y_, n_] := x - x + y + \text{Sum}[a3[n] \sin(n \pi x) \sin(n \pi y), \{n, 1, N\}]
\]
\[
\text{Plot3D}[u3[x, y, 100], \{x, 0, 1\}, \{y, 0, 1\}]
\]
Part II: To be completed in class.

Answer all questions. Show all work. No credit will be given for unsupported answers. The use of calculators or other electronic devices is NOT permitted for this part.
Place your answers on separate sheets. Use this question sheet as a cover sheet.

1. Solve the following initial-boundary value problem.

\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{9} \frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < 2, \quad 0 < t < \infty
\]
\[
u(0,t) = u(2,t) = 0, \quad 0 \leq t < \infty
\]
\[
u(x,0) = 2\sin(\pi x), \quad \frac{\partial u(x,0)}{\partial t} = -\sin(2\pi x), \quad 0 \leq x \leq 2
\]

(15 points)

2. Solve the following initial-boundary value problem.

\[
\frac{\partial u(x,t)}{\partial t} = 0.1 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 2, \quad 0 < t < \infty
\]
\[
u(0,t) = 0, \quad \nu(2,t) = 40, \quad 0 \leq t < \infty
\]
\[
u(x,0) = 4\left(5x + \sin(\pi x)\cos(\pi x)\right), \quad 0 \leq x \leq 2
\]

(15 points)

3. Consider the following Dirichlet problem for Laplace's equation.

\[
\nabla^2 u(x,y) = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \quad 0 < x < 2, \quad 0 < y < 2
\]
\[
u(2,y) = \cos\left(\frac{\pi y}{2}\right), \quad 0 \leq y \leq 2
\]
\[
u(x,2) = 1 - x, \quad 0 \leq x \leq 2
\]
\[
u(0,y) = \sin\left(\frac{\pi y}{4}\right), \quad 0 \leq y \leq 2
\]
\[
u(x,0) = \frac{x}{2}, \quad 0 \leq x \leq 2
\]

Find a solution of Laplace's equation, \(v(x,y)\), that takes on the same values as \(u(x,y)\) at the four vertices of the square.

(15 points)
1. \( u_{xx} = \frac{1}{l} u_{tt}, \quad 0 < x < 2, \quad 0 < t < \infty \) \\
\( c = 3, \quad l = 2 \) \\
wave equation, pinned string \\
\( u(0, t) = u(2, t) = 0, \quad 0 < t < \infty \) \\
\( u(x, 0) = 2 \sin \left( \frac{\pi x}{2} \right), \quad u_t(x, 0) = -\sin(2\pi x), \quad 0 \leq x \leq 2 \) \\
\( u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{n\pi x}{2} \right) + B_n \sin \left( \frac{n\pi x}{2} \right) \right) \sin \left( \frac{n\pi t}{2} \right) \) \\
\( u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{2} \right) = 2 \sin \left( \frac{\pi x}{2} \right), \quad 0 \leq x \leq 2 \). \quad \therefore A_2 = 2, \quad A_n = 0, \quad n \neq 2. \\
\( u_t(x, 0) = \sum_{n=1}^{\infty} 2n B_n \sin \left( \frac{n\pi x}{2} \right) = -\sin(2\pi x), \quad 0 \leq x \leq 2. \quad \therefore B_1 = -\frac{1}{4}, \quad B_n = 0, \quad n \neq 1. \) \\
\( u(x, t) = 2 \cos(3\pi t) \sin \left( \frac{\pi x}{2} \right) - \frac{1}{6} \sin(6\pi t) \sin(2\pi x) \)

2. \( u_t = 0.1 u_{xx}, \quad 0 < x < 2, \quad 0 < t < \infty \) \\
heat equation \\
\( l = 2, \quad k = 0.1 \) \\
constant temperature ends \\
\( u(0, t) = u(2, t) = 0, \quad 0 < t < \infty \) \\
\( u(x, 0) = 4 \left( 5x + \sin \pi x \cos \pi x \right), \quad 0 \leq x \leq 2 \) \\
Steady state equilibrium solution: \( \nu(x) = 20x, \quad 0 \leq x \leq 2 \) \\
Let \( W(x, t) = u(x, t) - \nu(x). \) Then: \\
\( W_t = 0.1 W_{xx}, \quad 0 < x < 2, \quad 0 < t < \infty \) \\
\( W(0, t) = W(2, t) = 0, \quad 0 < t < \infty \) \\
\( W(x, 0) = 4 \sin 2\pi x \cos \pi x = 2 \sin 2\pi x, \quad 0 \leq x \leq 2 \) \\
\( \therefore w(x, t) = \sum_{n=1}^{\infty} a_n e^{-\frac{n\pi^2}{4} t} \sin \left( \frac{n\pi x}{2} \right) \) \\
\( w(x, 0) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{2} \right) = 2 \sin 2\pi x, \quad 0 \leq x \leq 2 \). \quad \therefore a_1 = 2, \quad a_n = 0, \quad n \neq 1 \\
\( w(x, t) = 2 e^{-4\pi^2 t} \sin 2\pi x \) and \( u(x, t) = 2 e^{-4\pi^2 t} \sin 2\pi x + 20x \)

3. \\
\[ \begin{array}{ccc}
\sin(\pi x) & 1 - x & (2, 2) \\
\Delta u = 0 & \cos(\pi x) & \\
0 & 2 & x
\end{array} \] \\
Vertex values: \( u(0, 0) = 0 \) \\
\( u(2, 0) = 1 \) \\
\( u(2, 2) = -1 \) \\
\( u(0, 2) = 1 \) \\
\( \nu(0, 0) = \alpha_0 = 0 ; \nu(2, 2) = \alpha_0 + 2\alpha_1 = 1 \) \\
\( \nu(0, 2) = \alpha_0 + 2\alpha_2 = 2 \alpha_2 = 1 \) \\
\( \nu(2, 0) = \alpha_0 + 2\alpha_2 = 2 \alpha_2 = 1 \) \\
\( \nu(2, 2) = \alpha_0 + 2\alpha_2 + 4\alpha_3 = 2 + 4\alpha_3 = -1 \) and \( \alpha_3 = \frac{-3}{4} \) \\
\( \therefore \nu(x, y) = \frac{x}{2} + \frac{y}{2} - \frac{3}{4} xy \)

\( \Delta u = 0 \) \\
\( \cos(\pi x) \) \\
\( \Delta u = 0 \) \\
\( \sin(\pi x) \) \\
\( (2, 2) \)