Due: Thursday, April 17, at the beginning of class.

Use Mathematica as you wish for computations and for plotting. Provide complete documentation by appending your Mathematica notebook. Develop the solutions in a separate and complete exposition, referring the reader to the Mathematica work, transcribing your results and stating as needed that they were obtained using the software. DO NOT simply use Mathematica and expect the reader/graded to dig out your answers from your Mathematica output. The Mathematica notebook should serve as an appendix documenting your computations and exhibiting your plots. When creating Mathematica plots, use the PlotRange command as needed to properly display the graphs.

1. Consider the following initial-boundary value problem for the pinned string.
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{9} \frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < 2, \quad 0 < t < \infty
\]
\[
u(0,t) = u(2,t) = 0, \quad 0 \leq t < \infty
\]
\[
u(x,0) = U \sin^3 \left( \frac{\pi x}{2} \right), \quad \frac{\partial u(x,0)}{\partial t} = V \sin(\pi x), \quad 0 \leq x \leq 2
\]

(a) State the infinite series separation of variables solution for the pinned string problem and determine the series coefficients in terms of the constants \(U\) and \(V\).
(b) Assume that the instantaneous position and velocity at location \(x = \frac{1}{2}\) and time \(t = 2\) are found to be \(u(\frac{1}{2},2) = \frac{1}{2}\) and \(\frac{\partial u(\frac{1}{2},2)}{\partial t} = \sqrt{3}\). Determine the values of the constants \(U\) and \(V\).
(c) What is the maximum displacement attained by the string at its center as time evolves, i.e. what is the maximum value of \(|u(1,t)|\)?

2. Consider the following initial-boundary value problem for the pinned string.
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{9} \frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < 10, \quad 0 < t < \infty
\]
\[
u(0,t) = u(10,t) = 0, \quad 0 \leq t < \infty
\]
\[
u(x,0) = f(x) = \begin{cases} 
0, & 0 \leq x < 3 \\
3(x-3), & 3 \leq x \leq 4 \\
0, & 4 < x \leq 10
\end{cases}, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \leq x \leq 10
\]

(a) State the infinite series separation of variables solution for the pinned string problem and determine the series coefficients.
(b) Plot the solution at times \(t = 0\) and \(t = 3\) together on the same graph; use the first 200 terms of the series as an approximation to the solution. Indicate, by sketching arrows on the plot, the direction that the solution segments at time \(t = 3\) are travelling. (Note the persistence of the Gibbs phenomena at the later time.)
(c) Use Mathematica's Plot3D command to plot the (200 term approximate) solution surface over the subdomain \(0 \leq x \leq 10, \quad 0 \leq t \leq 6\). (Although the quality of the plot is somewhat ragged, it does illustrate the basic physics of multiple reflections with changes of polarity.)

Problem 3 on next page.
3. Consider the following insulated ends problem involving a nonhomogeneous heat equation.

\[
\frac{\partial u(x,t)}{\partial t} = 0.1 \frac{\partial^2 u(x,t)}{\partial x^2} + e^{-t}, \quad 0 < x < 1, \quad 0 < t < \infty
\]

\[
\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x} = 0, \quad 0 \leq t < \infty
\]

\[
u(x,0) = 4 \cos^2 \left(\frac{\pi x}{2}\right), \quad 0 \leq x \leq 1
\]

(a) Assume a solution of the form \( u(x,t) = \sum_{n=0}^{\infty} T_n(t) \cos(n\pi x) \).

Substitute this series solution into the nonhomogeneous heat equation, performing termwise partial differentiation. Equate coefficients of \( \cos(n\pi x) \) for each value of integer \( n = 0, 1, 2, 3, \ldots \) to obtain a family of first order differential equations for the \( T_n(t) \), \( n = 0, 1, 2, 3, \ldots \). Note that the differential equation for \( T_0(t) \) will be a (simple) nonhomogeneous equation; all others will be homogeneous equations. Obtain appropriate initial conditions from the coefficients of \( u(x,0) = 4 \cos^2 \left(\frac{\pi x}{2}\right) = 2 + 2 \cos(\pi x) \).

(b) Solve the initial value problems for the functions \( T_n(t) \), \( n = 0, 1, 2, 3, \ldots \) and form solution \( u(x,t) \).

(c) From your answer in part (b) determine the limit \( \lim_{t \to \infty} u(x,t) \) (assuming the limit exists).

(d) Compare your answer with that obtained in the following manner. Recall that \( E(t) = \int_0^1 u(x,t) \, dx \) is proportional to the energy content in the bar at time \( t \). From the nonhomogeneous heat equation one obtains:

\[
\frac{dE(t)}{dt} = \frac{d}{dt} \int_0^1 u(x,t) \, dx = \int_0^1 \frac{\partial u(x,t)}{\partial t} \, dx = \int_0^1 \left[ \frac{\partial^2 u(x,t)}{\partial x^2} + e^{-t} \right] \, dx = e^{-t}
\]

\[
E(0) = \int_0^1 u(x,0) \, dx = \int_0^1 \left[ 2 + 2 \cos(\pi x) \right] \, dx = 2
\]

Solve this simple problem for \( E(t) \) and use the fact that \( \lim_{t \to \infty} E(t) = u_\infty \cdot l = u_\infty \), where \( u_\infty = \lim_{t \to \infty} u(x,t) \).

Does the answer obtained in this way agree with that obtained in part (c)?
Math 4564 Solutions to H.W. #9  Spring 2014  W. Kohler

1. \( u_{xx} = \frac{1}{2} u_{tt}, \ 0 < x < 2, \ 0 < t < \infty \)

\( u(0,t) = u(2,t) = 0, \ 0 < t < \infty \)

\( u(x,0) = U \sin^{3}(\frac{\pi x}{2}), \ u_{x}(x,0) = V \sin(n \pi x), \ 0 < x < 2 \)

a) \( u(x,t) = \sum_{n=1}^{\infty} \left( A_{n} \cos(n \pi 3t) + B_{n} \sin(n \pi 3t) \right) \sin(n \frac{\pi x}{2}) \)

Note: \( \sin^{3}(\frac{\pi x}{2}) = \sin(n \pi x) \left( \frac{1}{2} - \frac{1}{2} \cos(n \pi x) \right) = \frac{1}{2} \sin(n \pi x) - \frac{1}{2} \sin(n \pi x) \cos(n \pi x) \)

\( = \frac{1}{4} \sin(n \pi x) - \frac{1}{4} \sin(n \pi x) \cos(n \pi x) \)

Impose initial conditions:

\( u(x,0) = \sum_{n=1}^{\infty} A_{n} \sin(n \frac{\pi x}{2}) = \frac{3}{4} U \sin(n \frac{\pi x}{2}) - \frac{3}{4} U \sin(3n \frac{\pi x}{2}) \)

\( \therefore A_{1} = \frac{3U}{4}, \ A_{3} = -\frac{U}{4}, \ A_{n} = 0, \ n \neq 1, 3 \)

\( u_{t}(x,0) = \sum_{n=1}^{\infty} 3n \pi B_{n} \sin(n \frac{\pi x}{2}) = V \sin(n \pi x) \)

\( \therefore 3n \pi B_{3} = V \text{ or } B_{3} = \frac{V}{3n \pi}, \ B_{n} = 0, \ n \neq 2 \).

\( \therefore u(x,t) = \frac{3}{4} U \cos(\frac{3n \pi t}{2}) \sin(n \frac{\pi x}{2}) - \frac{U \cos(n \frac{3 \pi t}{2}) \sin(n \frac{3 \pi x}{2})}{4} + \frac{V \sin(n \frac{3 \pi t}{2}) \sin(n \frac{\pi x}{2})}{4} \)

b) \( u(\frac{1}{3}, 2) = \frac{3}{4} U \cos(3 \pi \frac{2}{2}) \sin(\frac{n \pi}{2}) - \frac{U \cos(9 \pi \frac{2}{2}) \sin(\frac{n \pi}{2})}{4} + \frac{V \sin(6 \pi \frac{2}{2}) \sin(\frac{n \pi}{2})}{4} \)

\( = \frac{3}{4} U(-1)(\frac{1}{2}) - \frac{U(-1)(1)}{4} + 0 = \frac{1}{2} \)

\( \therefore (-\frac{3}{6} + \frac{1}{4}) U = \frac{1}{2} \quad \text{and} \quad U = -4 \)

\( u_{t}(\frac{1}{3}, 2) = -\frac{9n \pi U \sin(3n \pi \frac{2}{2}) \sin(\frac{n \pi}{2})}{8} + \frac{9n \pi U \sin(9n \pi \frac{2}{2}) \sin(\frac{n \pi}{2})}{8} + \frac{V \cos(6n \pi \frac{2}{2}) \sin(\frac{n \pi}{2})}{4} = \frac{13}{2} V = 13 \quad \Rightarrow \quad V = 2. \)

c) \( |u(1, t)| = \left| \frac{3}{4} U \cos(\frac{3n \pi t}{2}) + \frac{U \cos(9n \pi t)}{4} \right| = \left| -3 \cos(\frac{3n \pi t}{2}) + \cos(\frac{n \pi t}{2}) \right| \)

A maximum value of \( A \) is attained, at \( t = 0 \) for example.
2. \( u_{xx} = \frac{1}{9} u_{tt}, \quad 0 < x < 10, \quad 0 < t < \infty \quad c = 3, \quad l = 10 \)

\[ u(0, t) = u(10, t) = 0, \quad 0 < t < \infty \]

\[ u(x, 0) = \begin{cases} 
0, & 0 \leq x < 3 \\
\frac{3(x-3)}{3}, & 3 \leq x < 4 \\
0, & 4 < x \leq 10
\end{cases}, \quad u_t(x, 0) = 0 \]

(a) \( u(x, t) = \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{10} \right) \sin \left( \frac{n\pi t}{10} \right) \quad \text{The } B_n \text{ coefficients vanish since } u(t, 0) = 0. \)

\[ A_n = \frac{2}{10} \int_{0}^{10} f(x) \sin \left( \frac{n\pi x}{10} \right) dx = \frac{3}{5} \int_{3}^{4} (x-3) \sin \left( \frac{n\pi x}{10} \right) dx \]

\[ = -\frac{0}{n^2 \pi^2} \left( \sin(2n\pi) + 10(\sin(3n\pi) - \sin(2n\pi/5)) \right) \quad \text{(Mathematics)} \]

(b) See Mathematics for plot.

(c) See Mathematics for surface plot.

3. \( u_t = 0.1 u_{xx} + e^{-t} , \quad 0 < x < 1, \quad 0 < t < \infty \)

\[ u_x(0, t) = u_x(1, t) = 0, \quad 0 < t < \infty \]

\[ u(x, 0) = 4 \cos^2 \left( \frac{x}{2} \right) = 2 + 2 \cos(\pi x), \quad 0 \leq x \leq 1 \]

(a) Assume:

\[ u(x, t) = \sum_{n=0}^{\infty} T_n(t) \cos(n\pi x) \]

Substitute:

\[ u_t(x, t) = \sum_{n=0}^{\infty} T'_n(t) \cos(n\pi x) \]

\[ u_{xx}(x, t) = \sum_{n=0}^{\infty} T_n(t) (-n^2 \pi^2) \cos(n\pi x) \]

\[ \therefore \sum_{n=0}^{\infty} (T'_n(t) + 0.1(n\pi)^2 T_n(t)) \cos(n\pi x) = e^{-t} \]

and:

\[ T'_0(t) = -e^{-t} \]

\[ T'_n(t) + 0.1(n\pi)^2 T_n(t) = 0, \quad n \geq 1 \]

Impose initial condition: \( \sum_{n=0}^{\infty} T_n(0) \cos(n\pi x) = 2 + 2 \cos(\pi x), \quad 0 \leq x \leq 1 \)

\[ \therefore T_0(0) = 2, \quad T_i(0) = 2, \quad T_n(0) = 0, \quad n \geq 2 \]
3  b) The initial value problems become:

\[ T_0'(t) = e^{-t}, \quad T_0(0) = 2 \]

\[ T_1'(t) + 0.1 t^2 T_1(t) = 0, \quad T_1(0) = 2 \]

\[ T_n'(t) + 0.1 n^2 T_n(t) = 0, \quad T_n(0) = 0, \quad n \geq 2 \]

\[ T_0(t) = 2 + \int_0^t e^{-\lambda} d\lambda = 2 + (1 - e^{-t}) = 3 - e^{-t} \]

\[ T_1(t) = 2 e^{-0.1 t^2} \]

\[ T_n(t) = 0, \quad n \geq 2 \]

\[ u(x,t) = 3 - e^{-t} + 2 e^{-0.1 t^2} \cos(nx) \]

\[ \lim_{t \to \infty} u(x,t) = \lim_{t \to \infty} \left( 3 - e^{-t} + 2 e^{-0.1 t^2} \cos(nx) \right) = 3. \]

c) \[ \frac{dE}{dt} = \frac{d}{dt} \int_0^1 u(x,t) dx = \int_0^1 \frac{d}{dt} u(x,t) dx = \int_0^1 \left( 0.1 \frac{\partial^2 u(x,t)}{\partial x^2} + e^{-t} \right) dx \]

\[ = e^{-t} \quad \text{and} \quad E(0) = \int_0^1 (2 + 2 \cos(nx)) dx = 2 \]

\[ \frac{dE}{dt} = e^{-t}, \quad E(0) = 2 \quad \text{and} \quad E(t) = 2 + \int_0^t e^{-\lambda} d\lambda = 3 - e^{-t} \]

\[ \lim_{t \to \infty} E(t) = \lim_{t \to \infty} (3 - e^{-t}) = 3 = u_\infty(1) = u_\infty \]

\[ \lim_{t \to \infty} u(x,t) = u_\infty = 3 \] is obtained this way as well.
Problem 2:

(b):

\[
(3/5) \cdot \text{Integrate}[(x - 3) \cdot \sin[n \cdot \pi \cdot x / 10], \{x, 3, 4\}] \\
6 \left[ \frac{n \pi \cos \left( \frac{2n \pi}{5} \right) + 10 \left( \sin \left( \frac{3n \pi}{10} \right) - \sin \left( \frac{2n \pi}{5} \right) \right)}{n^2 \pi^2} \right] \\
A[n_] := -\frac{6 \left( n \pi \cos \left( \frac{2n \pi}{5} \right) + 10 \left( \sin \left( \frac{3n \pi}{10} \right) - \sin \left( \frac{2n \pi}{5} \right) \right) \right)}{n^2 \pi^2}; \\
u2[x_, t_, N_] := \text{Sum}[A[n] \cdot \cos[3 \cdot n \cdot \pi \cdot x / 10] \cdot \sin[n \cdot \pi \cdot x / 10], \{n, 1, N\}]; \\
\text{Plot}[[u2[x, 0, 200], u2[x, 3, 200]], \{x, 0, 10\}, \text{PlotRange} \rightarrow \{-3.5, 3.5\}]
Since the speed $c = 3$, the two halves of the initial displacement will have travelled a total distance of 9 units at time $t=3$. Therefore the two halves are travelling toward each other at that instant.

(c):

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Plot3D[u2[x, t, 200], {x, 0, 10}, {t, 0, 6}, PlotRange -> {-3, 3}]
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