Due: Thursday, April 10, at the beginning of class.

Use Mathematica as you wish for computations and for plotting. Provide complete documentation by appending your Mathematica notebook. Develop the solutions in a separate and complete exposition, referring the reader to the Mathematica work, transcribing your results and stating as needed that they were obtained using the software. DO NOT simply use Mathematica and expect the reader/ grader to dig out your answers from your Mathematica output. The Mathematica notebook should serve as an appendix documenting your computations and exhibiting your plots.

In problems 1(d) and 3, do not simply evaluate the infinite series numerically.

When creating Mathematica plots, use the PlotRange command as needed to properly display the graphs.

1. Consider the periodic function \( f: \mathbb{R} \to \mathbb{R} \), having period 2\( \pi \) and defined on \( -\pi \leq x \leq \pi \) as \( f(x) = (\pi - |x|)^2 \).
   (a) Does the function possess even or odd symmetry? Briefly explain your answer.
   (b) State the form of the Fourier series expansion for \( f \) and evaluate the coefficients. Exploit any symmetry properties to simplify the calculations.
   (c) Plot the function and its Fourier series representation on the same graph on the interval \( -\pi \leq x \leq \pi \), using the partial sum \( 0 \leq n \leq 100 \) as an approximation of the Fourier series.
   (d) Use the Fourier series obtained in (b) and your knowledge of the Fourier Convergence Theorem to sum the series \[ \sum_{n=1}^{\infty} \frac{1}{n^2}, \text{ i.e. } \sum_{n=1}^{\infty} \frac{1}{n^2} = ? \]

2. Consider the periodic function, \( f: \mathbb{R} \to \mathbb{R} \), having period 2 and defined on \( 0 \leq x < 2 \) as \( f(x) = \begin{cases} 2-x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases} \)
   (a) State the form of the Fourier series expansion for \( f \) and evaluate the coefficients.
   (b) Plot the Fourier series representation of the function on the interval \( -5 \leq x \leq 5 \), using the partial sum \( \frac{a_0}{2} + \sum_{n=1}^{100} ( ) \) as an approximation of the Fourier series.
   (c) To what value does the Fourier series converge to at \( x = 2 \)?

3. We have seen in class that the square wave and fully rectified sine wave have the following Fourier series expansions:

   (i) Square wave: \( f(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & 1 \leq x < 2 \end{cases} \), periodic with period 2; \( f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi x)}{(2k-1)} \).

   (ii) Fully-rectified sine wave: \( f(x) = |\sin(\pi x)| \), periodic with period 1; \( |\sin(\pi x)| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{4n^2-1} \).

   Use these Fourier series representations and your knowledge of the Fourier Convergence Theorem to sum the following two infinite series:

   (a) \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} \]
   (b) \[ \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \]

Problem 4 on the next page
4. Consider the following one-dimensional wave equation on an infinite spatial domain.

\[ \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 < t < \infty \]

(a) Solve two initial value problems consisting of the above equation and each of the two sets of initial conditions given below.

(b) In each case use Mathematica's Plot3D command to plot the solution surface on the subdomain 
\(-20 \leq x \leq 20, \quad 0 \leq t \leq 10\).

(i) \( u(x,0) = e^{-x^2}, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad -\infty < x < \infty \)

(ii) \( u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = e^{-x^2}, \quad -\infty < x < \infty \).

**Remark:** The solution of initial value problem (ii) will involve the Error function, defined by:

\[ Erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} \, dx. \]

This function is one of the functions in Mathematica's library (c.f. the Function Navigator in the Help menu.)
1. \( f: \mathbb{R} \to \mathbb{R}, \quad f(x) = (n-1)x \) on \(-\pi \leq x \leq \pi\) and periodic with period \(2\pi\).

a) \( f \) is an **even function** since \( f(-x) = (n-1)(-x) = (n-1)x = f(x) \).

b) \( f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \), a Fourier Cosine series with \( l = \pi \).

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} (\pi - x)^2 \cos(nx) \, dx = \frac{4(n\pi - \sin(n\pi))}{n^3 \pi} \quad \text{(Mathematica)}
\]

For integer \( n \geq 1 \), \( \sin(n\pi) = 0 \) and \( a_n = \frac{4\pi}{n^3} \), \( n = 1, 2, \ldots \)

\[
a_0 = \frac{2}{\pi} \int_{0}^{\pi} (\pi - x)^2 \, dx = \frac{2\pi^3}{3} \quad \text{(Mathematica)}.
\]

\[ f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} \]

\[ \therefore f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} \]

\[ \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]

\[ \quad \quad \quad \quad \text{See Mathematica for the two (virtually indistinguishable) plots.} \]

c) Since \( f \) is continuous everywhere, the Fourier Convergence Theorem asserts that the series converges to \( f(x) \) at all points. Therefore, at \( x = 0 \):

\[ (\pi - 0)^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{1}{4} (\pi^2 - \frac{\pi^2}{3}) = \frac{\pi^2}{6} \]

2. \( f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 2-x & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \end{cases} \), periodic with period \(2\).

\( 2l = 2 \quad \therefore l = 1 \) and:

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos(n\pi x) + b_n \sin(n\pi x) \right) \]

\[ a_n = \int_{0}^{2} f(x) \cos(n\pi x) \, dx = \int_{0}^{1} (2-x) \cos(n\pi x) \, dx + \int_{1}^{2} \cos(n\pi x) \, dx \]

\[ = \frac{1}{n\pi} \left( \frac{2\pi^2}{n^2} \right) \quad \text{(Mathematica)} \]

\[ b_n = \int_{0}^{2} f(x) \sin(n\pi x) \, dx = \int_{0}^{1} (2-x) \sin(n\pi x) \, dx + \int_{1}^{2} \sin(n\pi x) \, dx \]

\[ = -\frac{1}{n\pi} \left( \frac{2\pi^2}{n^2} \right) \quad \text{(Mathematica)} \]

\[ a_0 = \int_{0}^{2} f(x) \, dx = \frac{\pi}{2} \quad \text{(Mathematica)} \]

\[ \therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[ \frac{(1-(-1)^n)}{n^2} \cos(nx) + \frac{1}{n\pi} \sin(nx) \right] \]

b) See Mathematica for plot.
2. (c) $f(x)$ has a jump discontinuity at $x=2$.

The series converges to the average of the one-sided limits, i.e.,
\[ \frac{1}{2}(f(x^-) + f(x^+)) = \frac{1}{2} (2 + 1) = \frac{3}{2}. \]

\[ f(x) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi x)}{2k-1} \]

Note that $f$ is continuous at $x = \frac{1}{2}$.

\[ f(\frac{1}{2}) = 1 = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} \]

Note: \[ \sin((2k-1)\pi x) = \sin((k\pi - \frac{\pi}{2})x) = \sin(k\pi \cos x - \frac{\pi}{2} \sin x) = -\cos x \]

and:
\[ 1 = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} \Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4} \]

(ii) $f(x) = |\sin(mx)|$ periodic with period 1.

\[ f(x) = |\sin(mx)| = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{4n^2 - 1} \]

Since $f$ is continuous at $x = 0$:
\[ |\sin(0)| = 0 = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(0)}{4n^2 - 1} = \frac{\pi}{2} \]

4. The solution of the initial value problem:

\[ u_t(x,t) = u_{xx}(x,t), \quad -\infty < x < \infty, \quad 0 < t < \infty \quad (c = 1) \]

\[ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < \infty \]

is:
\[ u(x,t) = \frac{1}{2} \left( f(x-t) + f(x+t) \right) + \frac{1}{2} \int_{x-t}^{x+t} g(\lambda) \, d\lambda \]

For the given initial conditions:

a) $u(x,t) = \frac{1}{2} \left( e^{-(x-t)^2} + e^{-(x+t)^2} \right)$

b) $u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} e^\lambda \, d\lambda = \frac{1}{2\sqrt{\pi}} \left( \text{Erf}(t+x) + \text{Erf}(t-x) \right) - \frac{t^2}{4}$

b) See Mathematica for plots.
Problem 1(c):

\[ f_1[x_] := (\Pi - \text{Abs}[x])^2; \]

\[ (2/\Pi) \ast \text{Integrate}[f_1[x] \ast \cos[n \ast x], \{x, 0, \Pi\}] \]

\[ \frac{4(n \Pi - \sin[n \Pi])}{n^3 \Pi}; \]

\[ a_1[n_] := 4/n^2; \]

\[ a_1[0] = (2/\Pi) \ast \text{Integrate}[f_1[x], \{x, 0, \Pi\}] \]

\[ \frac{2\Pi^2}{3}; \]

\[ g_1[x_, n_] := a_1[0] / 2 + \text{Sum}[a_1[n] \ast \cos[n \ast x], \{n, 1, N\}]; \]

\[ \text{Plot}[[f_1[x], g_1[x, 100]], \{x, -\Pi, \Pi\}] \]

(d) Check:

\[ \text{Sum}[1/n^2, \{n, 1, \infty\}] \]

\[ \frac{\Pi^2}{6}; \]

Problem 2:

\[ f_2[x_] := 1 + (1-x) \ast (1-\text{UnitStep}[x-1]); \]
Check:

\[ \text{Plot}[f2[x], \{x, 0, 2\}, \text{PlotRange} \rightarrow \{0, 2\}] \]

\[
\begin{align*}
\text{Integrate}[f2[x] \times \cos[n \pi x], \{x, 0, 2\}] &= \\
&= \frac{1 - \cos[n \pi] + n \pi \sin[2 n \pi]}{n^2 \pi^2}
\end{align*}
\]

\[
\begin{align*}
\text{Integrate}[f2[x] \times \sin[n \pi x], \{x, 0, 2\}] &= \\
&= \frac{n \pi (-2 + \cos[2 n \pi]) + \sin[n \pi]}{n^2 \pi^2}
\end{align*}
\]

\[
\begin{align*}
a2[n_] := \frac{1 - \cos[n \pi] + n \pi \sin[2 n \pi]}{n^2 \pi^2}; \\
b2[n_] := -\frac{n \pi (-2 + \cos[2 n \pi]) + \sin[n \pi]}{n^2 \pi^2}; \\
a2[0] = \text{Integrate}[f2[x], \{x, 0, 2\}]
\end{align*}
\]

\[
\begin{align*}
a2[0] &= \frac{5}{2}; \\
g2[x_, N_] &= \text{Sum}[a2[n] \times \cos[n \pi x] + b2[n] \times \sin[n \pi x], \{n, 1, N\}] + \frac{5}{4};
\end{align*}
\]
Problem 3:

(a) Check:

\[ b(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} \]

\[ \text{Out[1]} = \frac{\pi}{4} \]

(b) Check:

\[ \sum_{n=1}^{\infty} \frac{1}{4n-1} \]

\[ \frac{1}{2} \]

Problem 4:

(a):

\[ u_4a[x_, t_] := (1/2) * (\text{Exp}[-(x-t)^2] + \text{Exp}[-(x+t)^2]) \]
Plot3D[u4a[x, t], {x, -20, 20}, {t, 0, 10}, PlotRange -> {0, 1.1}]

(b):

Integrate[Exp[-x^2], {x, t - x, x + t}]

\[
\frac{1}{2}\sqrt{\pi}\left(\text{Erf}[t-x] + \text{Erf}[t+x]\right)
\]

u4b[x_, t_] := \[\frac{1}{4}\sqrt{\pi}\left(\text{Erf}[t-x] + \text{Erf}[t+x]\right)];

Plot3D[u4b[x, t], {x, -20, 20}, {t, 0, 10}, PlotRange -> {0, 1}]