Due: Tuesday, November 4 at the beginning of class.

Use Mathematica as you wish for computations and for plotting. Provide complete documentation by appending your Mathematica notebook. Develop the solutions in a separate and complete exposition, referring the reader to the Mathematica work, transcribing your results and stating as needed that they were obtained using the software. DO NOT simply use Mathematica and expect the reader/grade to dig out your answers from your Mathematica output. The Mathematica notebook should serve as an appendix documenting your computations and exhibiting your plots.

1. (a) Solve the following initial-boundary value problem:

\[
4u_{xx}(x,t) = u_t(x,t), \quad 0 < x < 2, \quad 0 < t < \infty \\
u(0,t) = u(2,t) = 0, \quad 0 \leq t < \infty \\
u(x,0) = \sin^2\left(\frac{\pi x}{2}\right), \quad u_t(x,0) = -\sin(\pi x), \quad 0 \leq x \leq 2
\]

(b) Use Mathematica to plot the solution \( u(x,t) \) at times \( t = 0 \) and \( t = \frac{1}{3} \) on the same graph. Use as many terms in partial sum(s) as needed to obtain an adequate approximation.

2. (a) Solve the following initial-boundary value problem:

\[
4u_{xx}(x,t) = u_t(x,t), \quad 0 < x < 8, \quad 0 < t < \infty \\
u(0,t) = u(8,t) = 0, \quad 0 \leq t < \infty \\
u(x,0) = f(x) = \begin{cases} 
0, & 0 \leq x < 4 \\
x - 4, & 4 \leq x \leq 5 \\
0, & 5 < x \leq 8 
\end{cases}, \quad u_t(x,0) = 0, \quad 0 \leq x \leq 8
\]

(b) Use Mathematica to plot the solution \( u(x,t) \) at times \( t = 0 \) and \( t = 3 \) on the same graph. Use the PlotRange command as necessary to insure that your plot captures the entire solution. Mark with arrows (by hand) the direction that the two pieces of your solution are traveling at time \( t = 3 \).

3. (a) Solve the following initial-boundary value problem:

\[
4u_{xx}(x,t) = u_t(x,t), \quad 0 < x < 8, \quad 0 < t < \infty \\
u(0,t) = u(8,t) = 0, \quad 0 \leq t < \infty \\
u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 
0, & 0 \leq x < 4 \\
\sin(\pi(x - 4)), & 4 \leq x \leq 5 \\
0, & 5 < x \leq 8 
\end{cases}
\]

(b) Use Mathematica's Plot3D command to create two solution surface plots of \( u(x,t) \) on the spatial domain \( 0 \leq x \leq 8 \) and the two time intervals \( 0 \leq t \leq 4 \) and \( 4 \leq t \leq 8 \), respectively.

Continued on the next page.
4. Consider the following insulated ends initial-boundary value problem for the nonhomogeneous heat equation:

\[ u_t(x,t) = \kappa u_{xx}(x,t) + e^{-t}, \quad 0 < x < 2, \quad 0 < t < \infty \]
\[ u_x(0,t) = u_x(2,t) = 0, \quad 0 \leq t \leq \infty \]
\[ u(x,0) = 2\cos^2(\pi x) + \cos(2\pi x), \quad 0 \leq x \leq 2 \]

(a) Assume a solution of the form \( u(x,t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos\left(\frac{n\pi x}{2}\right), \quad 0 \leq x \leq 2, \quad 0 \leq t < \infty \).

Substitute this assumed form into the nonhomogeneous heat equation and use the orthogonality of the functions \( \{1, \cos\left(\frac{n\pi x}{2}\right)\}_{n=1}^{\infty} \) on the interval \( 0 \leq x \leq 2 \) to derive a system of initial value problems for the functions \( T_n(t), n = 0, 1, 2, 3, \ldots \).

(b) Solve the initial value problems for \( T_n(t), n = 0, 1, 2, 3, \ldots \) and form solution \( u(x,t) \).

(Hint: Your solution should reduce to the sum of two nonzero terms.)

(c) Determine \( \lim_{t \to \infty} u(x,t) \). Note that in this case the limit is no longer simply the average of the initial temperature distribution since the nonhomogeneous source term is injecting energy into the system.
1. \(4\phi_{xx}(x,t) = \phi_t(x,t), \quad 0 < x < 2, \quad 0 < t < \infty\) \hspace{1cm} \text{wave equation, pinned ends}

\[\phi(0,t) = \phi(2,t) = 0, \quad 0 < t < \infty\]

\[\phi(x,0) = \sin^2\left(\frac{\pi x}{2}\right) ; \quad \phi_t(x,0) = -\sin(\pi x), \quad 0 \leq x \leq 2\]

\[\phi(x,t) = \sum_{n=1}^{\infty} \left[ A_n \cos\left(n\pi t \right) + B_n \sin\left(n\pi t \right) \right] \sin\left(\frac{n\pi x}{2}\right)\]

\text{Imposing initial conditions:}

\[\phi(x,0) = \sin^2\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right), \quad 0 \leq x \leq 2\]

\[A_n = \frac{2}{\pi} \int_0^2 \sin^2\left(\frac{\pi x}{2}\right) \sin\left(\frac{n\pi x}{2}\right) \, dx = \frac{4(-1 + \cos(n\pi))}{(n^3 - 4n)n}, \quad n \neq 2 \text{ (Mathematica)}\]

\[A_n = \int_0^2 \sin^2\left(\frac{\pi x}{2}\right) \, dx = 0\]

\text{Note:} \quad A_n = \begin{cases} 0, & \text{if \(n\) even} \\ -\frac{8}{(n^3 - 4n)n}, & \text{if \(n\) odd} \end{cases}

\[\phi_t(x,0) = -\sin(\pi x) = \sum_{n=1}^{\infty} n\pi B_n \sin\left(\frac{n\pi x}{2}\right) = -\sin(\pi x), \quad 0 \leq x \leq 2\]

By inspection:

\[2\pi B_2 = -1, \quad B_2 = 0, \quad n \neq 2 \quad \Rightarrow \quad B_2 = -\frac{1}{2\pi}\]

Letting \(n = 2k - 1\):

\[\phi(x,t) = -\frac{8}{\pi} \sum_{k=1}^{\infty} \frac{\cos\left((2k-1)\pi t\right) \sin\left(\frac{(2k-1)\pi x}{2}\right) - \frac{1}{2} \sin(2\pi t) \sin(\pi x)}{(2k-1)^3 - 4(2k-1)}\]

b) See Mathematica for plot.

2. \(4\phi_{xx}(x,t) = \phi_t(x,t), \quad 0 < x < 8, \quad 0 < t < \infty\) \hspace{1cm} \text{wave equation}

\[\phi(0,t) = \phi(8,t) = 0, \quad 0 < t < \infty\]

\[\phi(x,0) = f(x) = \begin{cases} 0, & 0 \leq x \leq 4 \\ \sin(\pi(x-4)), & 4 \leq x \leq 5 \\ \phi_t(x,0) = 0, & 0 \leq x \leq 8 \\ 0, & 5 \leq x \leq 8 \end{cases}\]

Since \(\phi_t(x,0) = 0\):

\[\phi(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{8} - 2t\right) \sin\left(\frac{n\pi x}{8}\right)\]

\[A_n = \frac{2}{\pi} \int_0^4 f(x) \sin\left(\frac{n\pi x}{8}\right) \, dx = \frac{1}{4} \int_4^5 \sin\left(\pi(x-4)\right) \sin\left(\frac{n\pi x}{8}\right) \, dx\]

\[= \frac{-2 \left( \frac{\pi}{8} \cos\left(\frac{5n\pi}{8}\right) + 8 \sin\left(\frac{3n\pi}{8}\right) - 8 \sin\left(\frac{5n\pi}{8}\right) \right)}{n^2 \pi^2}, \quad n = 1, 2, 3, \ldots\]

(Mathematica)
2. (b) See Mathematica for plots. Since \( c = 2 \), the two pieces will have traveled 6 units of distance at \( t = 3 \). Therefore:

![Diagram](image)

3. \( 4u_{xx}(x,t) = u_{tt}(x,t), \ 0 < x < 8, \ 0 < t < 8 \) wave equation pinned ends
\( u(0,t) = u(8,t) = 0, \ 0 < t < \infty \)
\( u(x,0) = 0 \)
\( u_t(x,0) = g(x) = \begin{cases} 
0, & 0 \leq x < 4 \\
\sin \left( \pi (x - 4) \right), & 4 \leq x < 5 \\
0, & 5 \leq x \leq 8 
\end{cases} \)

(a)

Since \( u(x,0) = 0 \):
\[
u(x,t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{8} \right) \sin \left( \frac{n\pi t}{8} \right)
\]
and:
\[
u_t(x,0) = \sum_{n=1}^{\infty} \frac{m_n}{4} B_n \sin \left( \frac{n\pi x}{8} \right) = g(x), \ 0 \leq x \leq 8
\]
\[
B_n = \frac{4}{m_n^2} \int_{0}^{8} g(x) \sin \left( \frac{n\pi x}{8} \right) dx = \frac{1}{n\pi} \int_{4}^{5} \sin(n\pi(x-4)) \sin \left( \frac{n\pi x}{8} \right) dx = \begin{cases} 
-\frac{64\left( \sin \left( \frac{5\pi}{8} \right) + \sin \left( \frac{9\pi}{8} \right) \right)}{n\pi^2(4^2 - 4^2)}, & n \neq 8 \\
\frac{1}{16n}, & n = 8
\end{cases}
\]

b) See Mathematica for two surface plots.

Note that at \( t = 4 \), the solution \( u(x,t) \) is again zero. The polarity then switches and \( u(x,t) \leq 0 \) on the time interval \( 4 \leq t \leq 8 \).

4. \( u_t(x,t) = k u_{xx}(x,t) + e^{-t}, \ 0 < x < 2, \ 0 < t < \infty \) nonhomogeneous heat equation insulated ends
\( u_x(0,t) = u_x(2,t) = 0, \ 0 < t < \infty \)
\( u(x,0) = 2\cos^2(\pi x) = 1 + \cos(2\pi x), \ 0 \leq x \leq 2 \)

(a) Let \( u(x,t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t) \cos(\frac{n\pi x}{2}) \)

Substitute:
\[T'_0(t) + \sum_{n=1}^{\infty} T'_n(t) \cos(\frac{n\pi x}{2}) = k \sum_{n=1}^{\infty} (-T'_n(t) \left( \frac{n\pi}{2} \right)^2 \cos(\frac{n\pi x}{2}) + e^{-t}
\]
\[\cdot \left( T'_0(t) - e^{-t} \right) + \sum_{n=1}^{\infty} (-T'_n(t) + k \left( \frac{n\pi}{2} \right)^2 T_n(t)) \cos(\frac{n\pi x}{2}) = 0, \ 0 \leq x \leq 2, \ 0 < t < \infty -
\]
\[\therefore T'_0(t) - e^{-t} = 0
\]
\[T'_n(t) + k \left( \frac{n\pi}{2} \right)^2 T_n(t) = 0, \ n = 1, 2, 3, \ldots \] since \( \{1, \cos(\frac{n\pi x}{2})\}_{n=1}^{\infty} \) are orthogonal on \( 0 \leq x \leq 2 \).
4. (a) (cont.) Impose the initial condition.

\[ U(x, 0) = 1 + \cos(2\pi x) = T_0(0) + \sum_{n=1}^{\infty} T_n(0) \cos(\frac{2\pi n x}{L}), \quad 0 \leq x \leq L \]

Comparing terms: \( T_0(0) = 1, \quad T_4(0) = 0, \quad T_n(0) = 0, \quad n \neq 0, 4 \).

Summary: We obtain the initial value problems:

\[ T_0'(t) = e^{-t}, \quad T_0(0) = 1. \]
\[ T_4'(t) + k_4(2\pi)^2 T_4(t) = 0, \quad T_4(0) = 1, \quad 0 \leq t < \infty \]
\[ T_n'(t) + k_n(\frac{2\pi n}{L})^2 T_n(t) = 0, \quad T_n(0) = 0, \quad n \neq 0, 4 \]

The solutions are:

\[ T_0(t) = 1 + \int_t^0 e^{-\lambda} d\lambda = 1 + (1 - e^{-t}) = 2 - e^{-t} \]
\[ T_4(t) = e^{-4\pi^2 kt} \]
\[ T_n(t) = 0, \quad n \neq 0, 4. \]

\[ \therefore \quad U(x, t) = 2 - e^{-t} + \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2 k t}{L^2}} \cos(\frac{2\pi n x}{L}), \quad 0 \leq x \leq L, \quad 0 \leq t < \infty \]

(b) \( \lim_{t \to \infty} U(x, t) = 2 \)

Note that \( \frac{1}{2} \int_0^L U(x, 0) dx = 1 \) and \( \int_0^\infty e^{-t} dt = 1 \) combine to forming the limit.
Problem 1:

\[ 4u_{xx}(x,t) = u_t(x,t), \quad 0 < x < 2, \quad 0 < t < \infty \]
\[ u(0,t) = u(2,t) = 0, \quad 0 \leq t < \infty \]
\[ u(x,0) = \sin^2\left(\frac{\pi x}{2}\right), \quad u_t(x,0) = -\sin(\pi x), \quad 0 \leq x \leq 2 \]

(a):

\[
\int (\sin(\pi x / 2)^2) \sin(n \pi x / 2), \quad (x, 0, 2) \]

\[
\frac{4 (-1 + \cos(n \pi))}{(-4n + n^3) \pi}
\]

\[
\frac{4 (-1 + \cos(n \pi))}{(-4n + n^3) \pi}
\]

\[
\frac{4 (-1 + \cos(n \pi))}{(-4n + n^3) \pi}
\]

\[
\lim_{n \to 2} \frac{4 (-1 + \cos(n \pi))}{(-4n + n^3) \pi}
\]

0

\[ A1[k_1] := -(8 / \pi) / ((2 \ast k - 1) \ast 3 - 4 \ast (2 \ast k - 1)); \]

If one uses Mathematica to compute the B1 coefficients, the result is:

\[
\frac{1}{n \ast \pi} \int \sin^{n} \pi x \sin[n \pi x / 2], \quad (x, 0, 2) \]

This expression is 0 except when \( n = 2 \). In that case the quotient is indeterminate.

\[ B1[2] = \lim_{n \to 2} \frac{4 \sin[n \pi]}{n (-4 + n^2) \pi^2} \]

\[ 1 \]

\[ 2 \pi \]

\[ u1[x_, t, n_] := \]
\[ \text{Sum}[A1[k] \ast \cos[(2 \ast k - 1) \ast \pi \ast t] \ast \sin[(2 \ast k - 1) \ast \pi \ast x / 2], \quad (k, 1, n)] - \]
\[ (1 / (2 \ast \pi)) \ast \sin[2 \ast \pi \ast t] \ast \sin[\pi \ast x]; \]

(b):
Problem 2:

\[4u_{xx}(x,t) = u_{tt}(x,t), \quad 0 < x < 8, \quad 0 < t < \infty\]
\[u(0,t) = u(8,t) = 0, \quad 0 \leq t < \infty\]
\[u(x,0) = f(x), \quad u_t(x,0) = 0, \quad 0 \leq x \leq 8, \quad \text{where} \quad f(x) = \begin{cases} x-4 & 4 \leq x < 5 \\ 0 & 0 \leq x < 4 \text{ and } 5 \leq x \leq 8 \end{cases}\]

(a):
Initial displacement defined and plotted.

\[f2[x_] := (x-4)*(\text{UnitStep}[x-4]-\text{UnitStep}[x-5]);\]

\[\text{Plot}[f2[x], \{x, 0, 8\}]\]

\[(2/8) \cdot \text{Integrate}[f2[x] \cdot \sin[n \cdot \pi \cdot x/8], \{x, 4, 5\}]\]

\[\frac{1}{4} \int_4^5 f1[x] \sin \left(\frac{n \pi x}{8}\right) \, dx\]

\[A2[n_] := -\frac{2 \left(n \pi \cos \left(\frac{5n\pi}{8}\right) + 8 \sin \left(\frac{n\pi}{2}\right) - 8 \sin \left(\frac{5n\pi}{8}\right)\right)}{n^2 \pi^2};\]
u2[x_, t_, N_] := Sum[A2[n] * Cos[n * Pi * 2 * t / 8] * Sin[n * Pi * x / 8], {n, 1, N}];

Plot[ {u2[x, 0, 100], u2[x, 3, 100]}, {x, 0, 8}, PlotRange -> {-1, 1}]

Since c = 2, at time t = 3 the two pieces will have traveled 6 units of distance. They will have been reflected at the pins, changed polarity and be traveling toward each other.

Problem 3:

\(4u_{xx}(x, t) = u_t(x, t), \quad 0 < x < 8, \ 0 < t < \infty\)
\(u(0, t) = u(8, t) = 0, \ 0 \leq t < \infty\)
\(u(x, 0) = 0, \ u_t(x, 0) = g(x), \ 0 \leq x \leq 8, \ \text{where} \ g(x) = \begin{cases} 
\sin(\pi(x - 4)) & 4 \leq x < 5 \\
0 & 0 \leq x < 4 \ \text{and} \ 5 \leq x \leq 8
\end{cases}\)

\(g3[x_] := \sin[\pi * (x - 4)] * (\text{UnitStep}[x - 4] - \text{UnitStep}[x - 5]);\)

The initial velocity is defined and plotted.

Plot[g3[x], {x, 0, 8}]
\[(2 / (n * \pi * 2)) * \text{Integrate}[g3[x] * \sin[n * \pi * x / 8], \{x, 4, 5\}]\]
\[
64 \left( \sin\left[\frac{n \pi}{2}\right] + \sin\left[\frac{5n \pi}{8}\right] \right) \\
\frac{n (-64 + n^2)}{\pi^2}
\]

\[B3[n_] := -\frac{64 \left( \sin\left[\frac{n \pi}{2}\right] + \sin\left[\frac{5n \pi}{8}\right] \right)}{n (-64 + n^2) \pi^2};\]

\[B3[8] = \text{Limit}[B3[n], n \to 8]\]
\[
\frac{1}{16 \pi}
\]

\[u3[x_, t_, n_] := \text{Sum}[B3[n] * \sin[n * \pi * 2 * t / 8] * \sin[n * \pi * x / 8], \{n, 1, n\}];\]

To gain a better understanding of the surface plots that will be generated, we first plot the solution at some representative times.

\[\text{Plot}\left[\{u3[x, 0.5, 100], u3[x, 1, 100], u3[x, 2, 100]\}, \{x, 0, 8\}\right]\]

\[\text{Plot}\left[\{u3[x, 3.5, 100], u3[x, 4, 100], u3[x, 4.5, 100]\}, \{x, 0, 8\}, \text{PlotRange} \rightarrow \{-0.2, 0.2\}\right]\]

(b):
Plot3D[u3[x, t, 100], {x, 0, 8}, {t, 0, 4}]

Plot3D[u3[x, t, 100], {x, 0, 8}, {t, 4, 8}]

Plot3D[u3[x, t, 100], {x, 0, 8}, {t, 0, 4}]

Plot3D[u3[x, t, 100], {x, 0, 8}, {t, 4, 8}]}