Remarks:

1) The solution in the space-time region \( x \geq ct \) is the same as the infinite string solution since the influence of the pinned end at \( x = 0 \) has not had time to reach point \( x \).

2) The boundary at \( x = 0 \) lies in the space-time domain \( x \leq ct \). Therefore
\[
U(0,t) = \frac{1}{2} [f(ct) - f(-ct)] = 0, \quad 0 \leq t < \infty,
\]
as required.

3) The solution is continuous across the line \( x = ct \). Since \( f(0) = 0 \), both solutions reduce to \( \frac{1}{2} f(2ct) \).

An initial disturbance (with \( \phi = 0 \)) will propagate as shown.

The disturbance travelling to the left is reflected at the pinned end and propagates to the right with reversed polarity.

Finite Pinned String - Separation of Variables:

Consider now a string of length \( L \), pinned on both ends. The multiple reflections at both ends makes implementing the d’Alembert approach unwieldy for obtaining solutions over an appreciable time interval. Therefore, we resort to Separation of Variables.

Problem:
\[
U_{xx}(x,t) - c^2 U_{tt}(x,t) = 0, \quad 0 < x < L, \quad 0 < t < \infty
\]

Boundary conditions: \( U(x,0) = U(l,0) = 0 \), \( 0 < t < \infty \)

Initial conditions:
\[
U(x,0) = f(x), \quad U_t(x,0) = \phi(x), \quad 0 < x < L
\]
\[
(f(0) = f(l) = 0), \quad \phi(0) = \phi(l) = 0.
\]

Separation of Variables:

Look for solutions of the wave equation and also the homogeneous boundary conditions of the form:
\[
U(x,t) = X(x) T(t)
\]

Substituting:
\[
X''(x)T(t) - c^2X(x)T''(t) = 0
\]

or
\[
\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)} = -\sigma
\]

function of \( x \) only

function of \( t \) only

Separation equations: \( X''(x) + \sigma X(x) = 0, \quad 0 < x < l \)
\( T''(t) + \alpha^2 T(t) = 0, \quad 0 < t < \infty \)

**Impose the boundary conditions:**
\[ X(0)T(0) = 0 \quad \Rightarrow \quad X(0) = X(l) = 0 \]
\[ X(l)T(l) = 0, \quad 0 < t < \infty \]

We obtain a familiar two-point boundary value problem for \( X(x) \):
\[ X''(x) + \sigma X(x) = 0, \quad 0 < x < l \]
\[ X(0) = 0, \quad X(l) = 0 \]

This problem was encountered in the zero-temperature ends heat conduction problem.

\[ \mathcal{C}_n = \left( \frac{n\pi}{l} \right)^2 \]
\[ X_n(x) = \sin \left( \frac{n\pi x}{l} \right), \quad n = 1, 2, 3, \ldots \]

and:
\[ T_n''(t) + \left( \frac{n\sigma l}{\alpha} \right)^2 T_n(t) = 0, \quad t > 0 \]

\[ \therefore T_n(t) = A_n \cos \left( \frac{n\sigma l t}{\alpha} \right) + B_n \sin \left( \frac{n\sigma l t}{\alpha} \right), \quad n = 1, 2, 3, \ldots \]

where \( A_n, B_n \) are arbitrary constants.

\[ \therefore u_n(x, t) = \left( A_n \cos \left( \frac{n\sigma l t}{\alpha} \right) + B_n \sin \left( \frac{n\sigma l t}{\alpha} \right) \right) \sin \left( \frac{n\pi x}{l} \right) \]

and we form the solution template:
\[ u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{n\sigma l t}{\alpha} \right) + B_n \sin \left( \frac{n\sigma l t}{\alpha} \right) \right) \sin \left( \frac{n\pi x}{l} \right) \]

**Imposing the initial conditions:**

\[ u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{l} \right) = f(x), \quad 0 \leq x \leq l \quad (\text{Fourier Sine Series}) \]

\[ : \quad A_n = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{n\pi x}{l} \right) dx, \quad n = 1, 2, 3, \ldots \]

\[ u_t(x, t) = \sum_{n=1}^{\infty} \left( - \frac{n\sigma l}{\alpha} A_n \sin \left( \frac{n\pi x}{l} \right) + \frac{n\sigma l}{\alpha} B_n \cos \left( \frac{n\pi x}{l} \right) \right) \sin \left( \frac{n\pi x}{l} \right) \]

\[ : \quad u_t(x, 0) = \sum_{n=1}^{\infty} \frac{n\sigma l}{\alpha} B_n \sin \left( \frac{n\pi x}{l} \right) = g(x), \quad 0 \leq x \leq l \quad (\text{Fourier Sine Series}) \]

\[ : \quad \frac{n\sigma l}{\alpha} B_n = \frac{2}{l} \int_0^l g(x) \sin \left( \frac{n\pi x}{l} \right) dx, \quad n = 1, 2, 3, \ldots \]

or:
\[ B_n = \frac{2}{n\sigma l} \int_0^l g(x) \sin \left( \frac{n\pi x}{l} \right) dx, \quad n = 1, 2, 3, \ldots \]
Remark: (Travelling and Standing Waves)

The functions that arise in the d'Alembert approach, i.e. \( \phi(x-ct) \) and \( \psi(x+ct) \) are referred to as travelling waves. They retain their profile as they move with constant speed \( c \) to the right and left.

The functions that arise in the Separation of Variables approach, i.e. \( \cos \left( \frac{nmct}{\lambda} \right) \sin \left( \frac{nmx}{\lambda} \right) \) and \( \sin \left( \frac{nmct}{\lambda} \right) \sin \left( \frac{nmx}{\lambda} \right) \) are called standing waves.

Consider, for example, \( \cos \left( \frac{n\pi ct}{\lambda} \right) \sin \left( \frac{n\pi x}{\lambda} \right) \) (n=1 term). A sketch of this product is shown below. It is a spatial \( \sin \left( \frac{n\pi x}{\lambda} \right) \) multiplied by a time-varying amplitude \( \cos \left( \frac{n\pi ct}{\lambda} \right) \).

The Separation of Variables solution is a sum of standing waves. Note, however, that:

\[
\cos \left( \frac{n\pi ct}{\lambda} \right) \sin \left( \frac{n\pi x}{\lambda} \right) = \frac{1}{2} \left( \sin \left( \frac{n\pi x}{\lambda} (x+ct) \right) + \sin \left( \frac{n\pi x}{\lambda} (x-ct) \right) \right)
\]

\[
\sin \left( \frac{n\pi ct}{\lambda} \right) \sin \left( \frac{n\pi x}{\lambda} \right) = \frac{1}{2} \left( \cos \left( \frac{n\pi x}{\lambda} (x-ct) \right) - \cos \left( \frac{n\pi x}{\lambda} (x+ct) \right) \right)
\]

Therefore standing waves can be viewed as a superposition of travelling waves, moving in opposite directions.

Example:

\[
4u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 2, \quad 0 < t < \infty \quad \text{wave equation - pinned string}
\]

\[
u(0,t) = u(2,t) = 0, \quad 0 < x < 2
\]

\[
u(x,0) = \sin 2\pi x, \quad u_t(x,0) = -4\sin 4\pi x, \quad 0 < x < 2.
\]

\[
u(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{n\pi 2t}{\lambda} \right) + B_n \sin \left( \frac{n\pi 2t}{\lambda} \right) \right) \sin \left( \frac{n\pi x}{\lambda} \right)
\]

\[
u_t(x,t) = \sum_{n=1}^{\infty} \left( -n\pi A_n \sin \left( \frac{n\pi t}{\lambda} \right) + n\pi B_n \cos \left( \frac{n\pi t}{\lambda} \right) \right) \sin \left( \frac{n\pi x}{\lambda} \right)
\]

Imose initial conditions:

\[
u(x,0) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{\lambda} \right) = \sin 2\pi x, \quad 0 < x < 2
\]

\[
\therefore A_4 = 1, \quad A_n = 0, \quad n \neq 4
\]

\[
u_t(x,0) = \sum_{n=1}^{\infty} n\pi B_n \sin \left( \frac{n\pi x}{\lambda} \right) = -4\sin(4\pi x), \quad 0 < x < 2
\]

\[
\therefore 8\pi B_8 = -4, \quad B_8 = 0, \quad n \neq 8
\]

\[
u(x,t) = \cos \left( \frac{4\pi t}{\lambda} \sin \left( 2\pi x \right) - \frac{1}{2\pi} \sin \left( 8\pi t \right) \sin \left( 4\pi x \right)
\]

Note: \( \frac{1}{2} \left[ \sin \left( 2\pi (x-2t) \right) + \sin \left( 2\pi (x+2t) \right) \right] - \frac{1}{4\pi} \left[ \cos(4\pi (x-2t)) - \cos(4\pi (x+2t)) \right] \)
Example: This example will illustrate graphically that the Separation of Variables solution captures the physics of the wave equation problem exhibited by the d'Alembert solution.

Problem: \( u_{xx}(x,t) = u_{tt}(x,t), \quad 0 < x < \ell, \quad 0 < t < \infty \)
\( u(0,t) = u(\ell,t) = 0, \quad 0 \leq t < \infty \)
\( u(x,0) = \begin{cases} 
0, & 0 \leq x \leq \frac{\ell}{2} \\
4 \cos(\pi(x-1)), & \frac{\ell}{2} < x < \ell \\
0, & \ell \leq x \leq 6 
\end{cases} \)
\( u_t(x,0) = 0, \quad 0 \leq x \leq \ell \)

Wave equation pinned ends with \( c = 1, \quad l = \ell \). Since \( u_t(x,0) = 0, \quad 0 \leq x \leq \ell \), we expect the initial displacement to split into two pieces of amplitude 2 and travel in opposite directions. Upon reaching the pinned ends, they should be reflected with opposite polarity and start travelling in the opposite directions.

Since \( u_t(x,0) = 0, \quad 0 \leq x \leq \ell \), the Separation of Variables template has the form:

\[ u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{\ell}) \sin(\frac{n\pi}{\ell} t) \]

with:

\[ A_n = \frac{2}{\ell} \int_0^\ell u(x,0) \sin(\frac{n\pi}{\ell} x) dx = \frac{1}{3} \int_{\frac{\ell}{2}}^{\frac{3\ell}{2}} 4 \cos(\pi(x-1)) \sin(\frac{2\pi}{\ell} x) dx \]

Using Mathematica we obtain:

\[ A_n = \frac{-48 \left( \sin\left( \frac{5n\pi}{12} \right) + \sin\left( \frac{7n\pi}{12} \right) \right)}{n(n^2 - 3\ell)} \]

Note that the expression for \( A_n \) is indeterminate. Evaluating \( A_n \) explicitly:

\[ A_0 = \frac{1}{3} \int_{\frac{\ell}{2}}^{\frac{3\ell}{2}} 4 \cos(\pi(x-1)) \sin(\pi x) dx = 0 \]

Once \( A_n \) is defined we eliminate the problem caused by the indeterminacy by defining \( A_0 = 0 \).

Note that since \( u(x,0) \) is continuous everywhere, the Fourier Sine Series converges to \( u(x,0) \) at all \( x \). The 100 term partial sum forms a good approximation.

The solution plots on the next pages show the behavior we anticipated.
Wave equation - pinned ends with \( c = 1 \) and \( l = 6 \).

Problem:
\[
    u_{xx}(x,t) = u_{tt}(x,t), \quad 0 < x < 6, \quad 0 < t < \infty
\]
\[
    u(0,t) = u(6,t) = 0, \quad 0 < t < \infty
\]
\[
    u(x,0) = \begin{cases} 
        4 \cos(\pi(x-3)) & \frac{5}{2} \leq x \leq \frac{7}{2} \\
        0 & 0 \leq x < \frac{5}{2}, \quad \frac{7}{2} < x \leq 6
    \end{cases}
\]
\[
    u_t(x,0) = 0, \quad 0 \leq x \leq 6
\]

Plot of the initial displacement:

\[
f[x_] := 0 /; 0 \leq x < 5/2;
\]
\[
f[x_] := 4 \cos[\pi \ast (x - 3)] /; 5/2 \leq x \leq 7/2;
\]
\[
f[x_] := 0 /; 7/2 < x \leq 6;
\]
\[
\text{Plot}[f[x], \{x, 0, 6\}]
\]

Separation of Variables solution template:

\[
u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\frac{n \pi t}{6}) \sin(\frac{n \pi x}{6})
\]

\[
\frac{(1/3) \ast \text{Integrate}[4 \ast \cos[\pi \ast (x - 3)] \ast \sin[n \ast \pi \ast x / 6], \{x, 5/2, 7/2\}]}{48 \left( \sin\left[\frac{5\pi n}{12}\right] + \sin\left[\frac{7\pi n}{12}\right]\right)}
\]

\[
\frac{(-36 + n^2) \pi}{(\pi / 12) \pi}
\]

The indeterminacy when \( n = n = 6 \) is resolved by explicit integration.

\[
\frac{(1/3) \ast \text{Integrate}[4 \ast \cos[\pi \ast (x - 3)] \ast \sin[\pi \ast x], \{x, 5/2, 7/2\}]}{48 \left( \sin\left[\frac{5\pi n}{12}\right] + \sin\left[\frac{7\pi n}{12}\right]\right)}
\]

\[
0
\]

\[
A[n_] := -\frac{48 \left( \sin\left[\frac{5\pi n}{12}\right] + \sin\left[\frac{7\pi n}{12}\right]\right)}{(-36 + n^2) \pi};
\]

\[
A[6] = 0;
\]

\[
u[x_, t_, n_] := \text{Sum}[A[n] \ast \cos[n \ast \pi \ast t / 6] \ast \sin[n \ast \pi \ast x / 6], \{n, 1, N\}];
\]

The Separation of Variables solution is plotted at times \( t = 0, 1, 2 \) and \( 5 \), using a 100 term sum as a
approximation.

\[
\text{Plot}[u[x, 0, 100], \{x, 0, 6\}, \text{PlotRange} \rightarrow \{0, 4\}]
\]

\[
\text{Plot}[u[x, 1, 100], \{x, 0, 6\}, \text{PlotRange} \rightarrow \{0, 4\}]
\]

\[
\text{Plot}[u[x, 2, 100], \{x, 0, 6\}, \text{PlotRange} \rightarrow \{0, 4\}]
\]
Plot\left[u[x, 5, 100], \{x, 0, 6\}, \text{PlotRange} \to \{4, 0\}\right]

Plot of a solution surface showing the reflections from the pinned boundaries.

Plot3D[u[x, t, 100], \{x, 0, 6\}, \{t, 0, 20\}, \text{PlotRange} \to \{-5, 5\}\]
Nonhomogeneous Equations:

The Separation of Variables approach has some limited applicability in solving initial-boundary problems involving nonhomogeneous equations. We consider the zero temperature heat equation problem as an example.

**Problem:**
\[ u_t(x,t) - k u_{xx}(x,t) = F(x,t), \quad 0 < x < l, \quad 0 < t < \infty \]
\[ u(0,t) = u(l,t) = 0, \quad 0 \leq t \leq \infty \]
\[ u(x,0) = f(x), \quad 0 \leq x \leq l \]

For compatibility with the boundary conditions, we assume:

\[ F(0,t) = F(l,t), \quad 0 \leq t \leq \infty \quad \text{and} \quad F(0) = F(l) = 0. \]

\( F(x,t) \) models a source/sink of thermal energy.

**Solution:**

i) Expand \( F(x,t) \) in a Fourier Sine Series, viewing \( t \) as a parameter:

\[ F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin \left( \frac{n\pi x}{l} \right) \]

\[ F_n(t) = \frac{2}{l} \int_0^l F(x,t) \sin \left( \frac{n\pi x}{l} \right) dx, \quad n = 1, 2, 3, \ldots \]

Since \( F(x,t) \) is a known nonhomogeneous term, \( \{F_n(t)\}_{n=1}^{\infty} \) are known functions of time.

ii) Assume a solution of the form:

\[ u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \left( \frac{n\pi x}{l} \right) \]

Substituting into the nonhomogeneous equation:

\[ \sum_{n=1}^{\infty} T_n(t) \sin \left( \frac{n\pi x}{l} \right) - k \sum_{n=1}^{\infty} \left( \frac{n\pi}{l} \right)^2 T_n(t) \sin \left( \frac{n\pi x}{l} \right) = \sum_{n=1}^{\infty} F_n(t) \sin \left( \frac{n\pi x}{l} \right) \]

or:

\[ \sum_{n=1}^{\infty} \left[ T_n'(t) + k \left( \frac{n\pi}{l} \right)^2 T_n(t) - F_n(t) \right] \sin \left( \frac{n\pi x}{l} \right) = 0, \quad 0 \leq x \leq l \]

Therefore:

\[ T_n'(t) + k \left( \frac{n\pi}{l} \right)^2 T_n(t) = F_n(t), \quad 0 < t < \infty, \quad n = 1, 2, 3, \ldots \]

Impose initial condition:
\[ u(x,0) = f(x) = \sum_{n=1}^{\infty} T_n(0) \sin \left( \frac{n\pi x}{l} \right), \quad 0 \leq x \leq l \]

\[ T_n(0) = 2 \int_0^l f(x) \sin \left( \frac{m\pi x}{l} \right) dx = f_n, \quad n=1,2,3,\ldots \]

**Summary:** We obtain an infinite collection of initial value problems for \( \{T_n(t)\}_{n=1}^{\infty} \):

\[ T_n'(t) + \kappa \left( \frac{m\pi^2}{l^2} \right) T_n(t) = F_n(t), \quad 0 < t < \infty \]

\[ T_n(0) = f_n, \quad n=1,2,3,\ldots \]

**Example:** Solve: \( u_t(x,t) - \kappa u_{xx}(x,t) = \epsilon^t \sin(\eta x), \quad 0 < x < 2, \quad 0 < t < \infty \)

\[ u(0,t) = u(2,t) = 0, \quad 0 < t < \infty \]

\[ u(x,0) = 20 \sin 2\pi x, \quad 0 \leq x \leq 2 \]

**Assume:**

\[ u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \left( \frac{m\pi x}{l} \right) \]

Since \( F_n(x,t) = \epsilon^t \sin(\eta x) = \sum_{n=1}^{\infty} F_n(t) \sin \left( \frac{m\pi x}{l} \right) \), we have:

\[ F_2(t) = \epsilon^t, \quad F_n(t) = 0, \quad n \neq 2, \quad 0 < t < \infty \]

Also: \( f(x) = 20 \sin 2\pi x = \sum_{n=1}^{\infty} f_n \sin \left( \frac{m\pi x}{l} \right) \Rightarrow f_2 = 20, \quad f_n = 0, \quad n \neq 4 \)

Therefore, the initial value problems for \( \{T_n(t)\}_{n=1}^{\infty} \) become:

\[ T_n'(t) + \kappa \left( \frac{m\pi^2}{l^2} \right) T_n(t) = 0, \quad T_n(0) = 0, \quad n \neq 2,4 \]

\[ T_2'(t) + \kappa \left( \frac{4\pi^2}{l^2} \right) T_2(t) = \epsilon^t, \quad T_2(0) = 0 \]

\[ T_4'(t) + \kappa \left( \frac{16\pi^2}{l^2} \right) T_4(t) = 0, \quad T_4(0) = 20 \]

The solutions are:

1) \( T_n(t) = 0, \quad n \neq 2,4 \)

2) \( \left( e^{\frac{4\pi^2}{l^2} t} T_2(t) \right)' = e^t \left( \frac{4\pi^2}{l^2} \right) T_2(t) = \frac{e^t}{\frac{4\pi^2}{l^2}} + C_2 \)

   (assuming \( \kappa \left( \frac{4\pi^2}{l^2} \right) + 1 \) )

   \[ T_2(t) = \frac{e^{-t}}{\frac{4\pi^2}{l^2}} + C_2 e^{4\pi^2 t} \]

   \[ T_2(0) = 0 \Rightarrow C_2 = -\frac{1}{\frac{4\pi^2}{l^2}} \quad \text{and} \quad T_2(t) = \frac{e^{-t} - e^{4\pi^2 t}}{\frac{4\pi^2}{l^2}} \]

3) \( e^{\frac{4\pi^2}{l^2} t} T_4(t)' = 0 \Rightarrow T_4(t) = C_4 e^{-4\pi^2 t} \quad T_4(0) = C_4 = 20 \quad \therefore T_4(t) = 20 e^{-4\pi^2 t} \)
Solution:

\[ u(x,t) = \left( \frac{2}{k\pi^2 - 1} \right) \sin(\pi x) + 20 \varepsilon \sin(2\pi x) \]

**Laplace’s Equation:**

Laplace’s equation (and the nonhomogeneous version called Poisson’s equation) arises in describing potential functions associated with conservative vector fields. It also arises in describing steady state heat flow.

**Examples:**

i) The electrostatic field \( \varepsilon \) satisfies \( \nabla \times \varepsilon = 0 \). From this, one can show that \( \varepsilon = -\nabla \phi \), the (negative) gradient of a scalar potential. In the absence of charge, we also have \( \nabla \cdot \varepsilon = 0 \). Therefore:

\[ \nabla \cdot (-\nabla \phi) = 0 \quad \text{or} \quad \nabla^2 \phi = \Delta \phi = 0. \]

ii) If the velocity field of a fluid is irrotational and the fluid is incompressible, i.e. \( \nabla \times \mathbf{u} = 0 \) and \( \nabla \cdot \mathbf{u} = 0 \), Laplace’s equation emerges, i.e. \( \mathbf{u} = -\nabla \phi \) and \( \Delta \phi = 0 \).

iii) Steady-state heat flow:

Generally, \( u_t(x,t) = k \Delta u(x,t) \), but if \( u_t = 0 \), i.e. the temperature has equilibrated to a steady state distribution, then:

\( \Delta u(x) = 0 \).

**Remarks:**

In contrast to the heat and wave equations, the initial value problem is not appropriate. Rather, we consider a boundary value problem where data is specified on the boundary of a domain and the problem is to determine the potential that takes on this boundary data.

Typically, the values of \( \phi \) or its normal derivative \( \frac{\partial \phi}{\partial n} = \nabla \phi \cdot n \) are specified on boundary \( \partial \Omega \).

In the steady state heat flow problem, this amounts to specifying either the temperature or the flux on boundary \( \partial \Omega \).
The problem where $\Delta \phi = 0$ in domain $\Omega$ and $\phi = f$ on $\partial \Omega$ is called the Dirichlet problem.

If $\Delta \phi$ is specified on $\partial \Omega$, the problem is called the Neumann problem.

We will consider the Dirichlet problem.

ii) Solutions of Laplace's equation satisfy a Maximum (and Minimum) principle. Let $U(x,y)$ satisfy $\Delta U(x,y) = 0$ in domain $\Omega$. Unless $U(x,y)$ is constant in $\overline{\Omega} = \Omega \cup \partial \Omega$, $U$ takes on its maximum and minimum values on the boundary $\partial \Omega$.

To see why this is plausible, assume, for example, that solution $U$ has a local maximum at interior point $(x_0, y_0)$. Then:

$$U_{xx}(x_0, y_0) < 0, \quad U_{yy}(x_0, y_0) < 0 \quad \text{(concave down)},$$

However, Laplace's equation requires

$$U_{xx}(x_0, y_0) + U_{yy}(x_0, y_0) = 0.$$

Interior saddle points are possible.

**Dirichlet Problem for the Rectangle:**

The problem to be solved is $\Delta U(x,y) = 0$ in $\Omega = \{(x,y): 0 \leq x \leq a, \quad 0 \leq y \leq b\}$

with $U = f$ on $\partial \Omega$. We assume that $f$ varies continuously on $\partial \Omega$.

We begin by considering special cases, where non-zero boundary data is specified on only one side of the boundary. These special case solutions will then be used to solve the general case.

**Problem (special case):**

$$\Delta U(x,y) = U_{xx}(x,y) + U_{yy}(x,y) = 0, \quad 0 < x < a, \quad 0 < y < b$$

**Boundary conditions:**

$u(x,0) = 0, \quad 0 \leq x \leq a$

$u(x,b) = 0, \quad 0 \leq x \leq a$

$u(0,y) = 0, \quad 0 \leq y \leq b$

$u(a,y) = f(y), \quad 0 \leq y \leq b$

with $f(a) = f(b) = 0$.

**Separation of Variables:**

We look for non-trivial solutions of Laplace's equation,

$u(x,y) = X(x)Y(y)$

that also satisfy the 3 zero boundary conditions.
Substituting:
\[ u_{xx}(x,y) + u_{yy}(x,y) = X''(x)Y(y) + X(x)Y''(y) = 0 \]

\[ \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \sigma \] 

And:
\[ X''(x) - \sigma X(x) = 0, \quad 0 < x < a \]
\[ Y''(y) + \sigma Y(y) = 0, \quad 0 < y < b \]

Imposing the 3 homogeneous boundary conditions:

\[ X(a)Y(a) = 0, \quad 0 \leq x \leq a \] 

To obtain nontrivial solutions, we need:
\[ X(a) = 0, \quad Y(a) = Y(b) = 0 \]

Summary:

i) \[ X''(x) - \sigma X(x) = 0, \quad 0 < x < a \]
\[ X(0) = 0 \]

ii) \[ Y''(y) + \sigma Y(y) = 0, \quad 0 < y < b \]
\[ Y(0) = Y(b) = 0 \]

The 2-point boundary value problem for \( Y(y) \) is one that we have seen before (zero temperature ends and pinned string problems). We obtain:
\[ \sigma_n = \left( \frac{nm}{b} \right)^2, \quad Y_n(y) = \sin \left( \frac{nm}{b} y \right); \quad n=1,2,3,\ldots \]

\( X_n(x) \) problem becomes:
\[ X''(x) - \left( \frac{nm}{b} \right)^2 X_n(x) = 0, \quad 0 < x < a \]
\[ X_n(0) = 0 \]

\[ X_n(a) = k_1 e^{\frac{nm}{b} a} + k_2 e^{-\frac{nm}{b} a} \]

The general solution of the differential equation has the form:
\[ X_n(x) = k_1 e^{\frac{nm}{b} x} + k_2 e^{-\frac{nm}{b} x} \]

To satisfy the boundary condition \( X_n(0) = 0 \), it is convenient to take \( k_1 = \frac{1}{2}, \quad k_2 = -\frac{1}{2} \) and:
\[ X_n(x) = \frac{1}{2} \left( e^{\frac{nm}{b} x} - e^{-\frac{nm}{b} x} \right) = \sinh \left( \frac{nm}{b} x \right); \quad n=1,2,3,\ldots \]

The "building block" solutions obtained using separation of variables are:
\[ u_n(x,y) = \sinh \left( \frac{nm}{b} x \right) \sin \left( \frac{nm}{b} y \right); \quad n=1,2,3,\ldots \]

Remark: It is important to note that \( \sinh(x) = 0 \) only at \( x = 0 \). It is an odd function vanishing only at the origin.
We look for a solution of the Dirichlet problem of the form:

\[ u(x, y) = \sum_{n=1}^{\infty} a_n \sinh(\frac{n\pi x}{b}) \sin(\frac{n\pi y}{b}) \]

The solution must satisfy the remaining boundary condition:

\[ u(a, y) = \sum_{n=1}^{\infty} a_n \sinh(\frac{n\pi a}{b}) \sin(\frac{n\pi y}{b}) = f(y), \quad 0 \leq y \leq b \]

This is a Fourier Sine Series. We have:

\[ a_n = \frac{2}{b} \int_0^b f(y) \sin(\frac{n\pi y}{b}) \, dy, \quad n = 1, 2, 3, \ldots \]

Since \( \frac{n\pi a}{b} \neq 0 \), \( \sinh(\frac{n\pi a}{b}) \neq 0 \) and we have:

\[ a_n = \frac{2}{b \sinh(\frac{n\pi a}{b})} \int_0^b f(y) \sin(\frac{n\pi y}{b}) \, dy, \quad n = 1, 2, 3, \ldots \]

**Example:** Solve the Dirichlet problem sketched below.

\[
\begin{array}{c|c|c|c|c}
\hline
x & u(x, 0) = 0 & u(x, 1) & \Delta u(x, y) = 0, \quad 0 < x, y < 1 \\
\hline
y & u(0, y) = 0 & \Delta u(1, y) = 2 \sin 3\pi y & u(0, y) = 0, \quad 0 < y < 1 \\
\hline
\end{array}
\]

Solution:

\[ u(x, y) = \sum_{n=1}^{\infty} a_n \sinh(\frac{n\pi x}{b}) \sin(\frac{n\pi y}{b}) \]

Imposing the nonhomogeneous boundary condition:

\[ u(1, y) = \sum_{n=1}^{\infty} a_n \sinh(\frac{n\pi x}{b}) \sin(\frac{n\pi y}{b}) = 2 \sin 3\pi y, \quad 0 \leq y \leq 1 \]

By inspection:

\[ a_3 \sinh(3\pi) = 2, \quad a_n = 0, \quad n \neq 2 \]

and:

\[ u(x, y) = \frac{2 \sinh(3\pi x) \sin(3\pi y)}{\sinh(3\pi)} \]

The solution surface is shown on the next page. Note that the maximum and minimum values of \( u(x, y) \) occur on the boundary.
math 4564  Dirichlet problem example

Problem:
\[ \Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad 0 < x < 1, \quad 0 < y < 1 \]
u(x,0) = u(x,1) = 0, \quad 0 \leq x \leq 1
u(0,y) = 0, \quad 0 \leq y \leq 1
u(1,y) = 2\sin(3\pi y), \quad 0 \leq y \leq 1

Solution:  \[ u(x,y) = \frac{2\sinh(3\pi x)\sin(3\pi y)}{\sinh(3\pi)} \]

Plot3D[2*Sin[3*Pi*x]*Sin[3*Pi*y]/Sinh[3*Pi],
{x, 0, 1}, {y, 0, 1}, PlotRange -> {-2.1, 2.1}]