The Delta Function and Impulsive Response:

The delta function is an idealization, a device used to model point sources and impulsive forces.

To motivate the delta function definition, consider a short pulse of duration $\epsilon$ and amplitude $1/\epsilon$ (unit strength).

\[
P_\epsilon(t-t_0) = \begin{cases} \frac{1}{\epsilon}, & t_0 \leq t < t_0 + \epsilon \\ 0, & \text{otherwise} \end{cases}
\]

Graph of $p_\epsilon(t-t_0)$

Let $f(t)$ be continuous on $0 \leq t < \infty$. Then, applying the Integral Theorem of the Mean:

\[
\int_0^\infty f(t) p_\epsilon(t-t_0) \, dt = \int_{t_0}^{t_0+\epsilon} f(t) \frac{1}{\epsilon} \, dt = \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} f(t) \, dt = f(\xi), \text{where } t_0 < \xi < t_0 + \epsilon
\]

If we now let $\epsilon \downarrow 0$, $\xi \uparrow t_0$ and:

\[
\lim_{\epsilon \downarrow 0} \int_0^\infty f(t) p_\epsilon(t-t_0) \, dt = \lim_{\xi \uparrow t_0} f(\xi) = f(t_0).
\]

Using these heuristics as motivation, we define the delta function as:

**Def:** \[
\int_a^b f(t) \delta(t-t_0) \, dt = \begin{cases} f(t_0), & a \leq t_0 < b \\ 0, & \text{otherwise} \end{cases}
\]

Heuristically, we think of the delta function as a pulse of zero duration, infinite amplitude and unit strength. (It is made rigorous in the theory of distributions.)

**Note:** \[
\int_0^t \delta(t-t_0) \, d\lambda = \begin{cases} 0, & t < t_0 \\ f, & t \geq t_0 \end{cases}
\]

\[\therefore \int_0^t \delta(t-t_0) \, d\lambda = \mathcal{H}(t-t_0), \text{ the Heaviside function.}\]
Laplace Transform: Using the definitions:

\[ \mathcal{L}(\delta(t-t_0)) = \int_0^\infty \delta(t-t_0)e^{-st} \, dt = e^{-st_0}, \ t_0 > 0. \]

In particular,

\[ \mathcal{L}(\delta(t)) = 1 \]

More heuristics:

It's a useful fiction to think of the delta function as the derivative of the unit step function, \( h(t) \).

Consider a mollified version of the Heaviside function, i.e., a smoothed version similar to that sketched below that is actually differentiable. The derivative would be the short duration spike shown, a smooth approximation to the delta function.

\[ \begin{array}{c}
\text{Smoothed step function} \\
\text{and its derivative.}
\end{array} \]

The fiction is also useful in helping one anticipate the behavior of solutions. As an example, we will consider an initial value problem of the form:

\[ \ddot{y}(t) + \alpha \dot{y}(t) + \beta y(t) = \delta(t-t_0), \ y(0) = y_0, \ y'(0) = y'_0 \]

modelling a linear system subjected to an impulsive "ping" at \( t = t_0 \).

One reason that the most ill-behaved term on the left, \( \ddot{y} \), has "delta function behavior" at \( t = t_0 \). Therefore, \( y' \) should undergo a jump discontinuity at \( t = t_0 \) and \( y \) should be continuous at \( t = t_0 \).

We will now consider an illustrative example.

Example: Solve: \[ \dddot{y}(t) + 3\ddot{y}(t) + 2\dot{y}(t) = \delta(t-2), \ y(0) = 1, \ y'(0) = 0 \]

One could interpret this problem as modeling an overdamped mass-spring-dashpot system having unit initial displacement, zero initial velocity and subjected to an impulsive "ping" at \( t = 2 \).

Let \( Y(s) = \mathcal{L}(y(t)) \). Then:

\[ s^2 Y(s) - s(1) - 0 + 3(sY(s) - 1) + 2Y(s) = e^{-2s} \]

\[ (s^2 + 3s + 2)Y(s) - s - 3 = e^{-2s} \Rightarrow Y(s) = \frac{s + 3}{(s+1)(s+2)} + \frac{e^{-2s}}{(s+1)(s+2)} \]

\[ \Rightarrow \mathcal{L}(y(t)) = \frac{s + 3}{(s+1)(s+2)} + \frac{e^{-2s}}{(s+1)(s+2)} \]
Note: \[ \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2} \quad \text{and} \quad \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} \]

\[ Y(s) = \frac{2}{s+1} - \frac{1}{s+2} + e^{-2s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) \]  

Using the 2nd Shifting Theorem:

and:

\[ y(t) = 2e^{-t} - e^{-2t} + (e^{-(t-2)} - e^{-(t-2)})h(t-2) \]

Therefore:

\[ y(t) = \begin{cases} 
2e^t - e^{2t}, & 0 < t < 2 \\
2e^t - e^{2t} + e^{-(t-2)} - e^{-(t-2)}, & 2 \leq t < \infty 
\end{cases} \]

Note that \( y(t) \) is continuous at \( t = 2 \) since \( \lim_{t \to 2} (e^{-(t-2)} - e^{-(t-2)}) = 0 \).

The next page illustrates the solution using Mathematica. As anticipated, the solution is continuous with a discontinuous derivative at \( t = 2 \).

Note that Mathematica has adopted the heuristics we discussed, i.e., that \( \frac{dB}{dt}(t-2) = B(t-2) \).

Example: Solve: \[ y'(t) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ B(t-2) \end{bmatrix}, \quad y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

What to expect? In component form: \[ y_1'(t) = -2y_1(t) + y_2(t), \quad y_1(0) = 2 \]

\[ y_2'(t) = -y_2(t) + B(t-2), \quad y_2(0) = 1 \]

This system could be solved recursively. First solve the second equation for \( y_1(t) \). Then substitute this known expression into the first equation and solve for \( y_2(t) \).

Look at this way, we expect \( y(t) \) to have jump discontinuity and \( y_1(t) \) to be continuous with derivative that has a jump discontinuity at \( t = 2 \).

Let \( Y(s) = L(y(t)) \) and \( A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \). Then:

\[ sY(s) - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = AY(s) + \begin{bmatrix} 0 \\ e^{-2s} \end{bmatrix} \]

\[ \therefore Y(s) = (sI - A)^{-1}\begin{bmatrix} 2 \\ 1 + e^{-2s} \end{bmatrix} \]

\[ (sI - A) = \begin{bmatrix} s+2 & -1 \\ 0 & s+1 \end{bmatrix} \Rightarrow (sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix} \]

\[ \therefore Y(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 + e^{-2s} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+2} + \frac{1+e^{-2s}}{(s+1)(s+2)} \\ \frac{1+e^{-2s}}{s+1} \end{bmatrix} \]
Consider \( y''(t) + 3y'(t) + 2y(t) = \delta(t-2), \, y(0) = 1, \, y'(0) = 0. \) 

\[
\text{DSolve}\{y''[t] + 3y'[t] + 2y[t] = \text{DiracDelta}[t - 2], \, y[0] = 1, \, y'[0] = 0, \, y[t], \, t\}
\]

\[
\{y[t] \rightarrow e^{-2t} (\text{HeavisideTheta}[-2 + t] + e^{4t} \text{HeavisideTheta}[-2 + t])\}
\]

\[
y[t_] := e^{-2t} (-1 + 2 e^t - e^4 \text{HeavisideTheta}[-2 + t] + e^{2t} \text{HeavisideTheta}[-2 + t]);
\]

\[
\text{Plot}[y[t], \{t, 0, 5\}]
\]

\[
yprime[t_] := D[y[t], t];
\]

\[
yprime[t]
\]

\[
e^{-2t} (2 e^t - e^4 \text{DiracDelta}[-2 + t] + e^{2t} \text{DiracDelta}[-2 + t] + e^{2t} \text{HeavisideTheta}[-2 + t]) - 2 e^{-2t} (-1 + 2 e^t - e^4 \text{HeavisideTheta}[-2 + t] + e^{2t} \text{HeavisideTheta}[-2 + t])
\]

\[
\text{Simplify}\[
\]

\[
e^{-2t} (2 e^t - e^4 \text{DiracDelta}[-2 + t] + e^{2t} \text{DiracDelta}[-2 + t] + e^{2t} \text{HeavisideTheta}[-2 + t]) - 2 e^{-2t} (-1 + 2 e^t - e^4 \text{HeavisideTheta}[-2 + t] + e^{2t} \text{HeavisideTheta}[-2 + t])
\]

\[
e^{-2t} (2 - 2 e^t + (2 e^t - e^{2t}) \text{HeavisideTheta}[-2 + t])
\]
Plot\[e^{-2t} (2 - 2 e^t + (2 e^4 - e^{2t}) \text{HeavisideTheta[-2 + t]}), (t, 0, 5)\]

\[y2prime[t_] := D[yprime[t], t];\]

\[y2prime[t] = \]
\[-4 \ e^{-2t} \]
\[\{2 e^t - e^4 \text{DiracDelta[-2 + t]} + e^{2t} \text{DiracDelta[-2 + t]} + e^{2t} \text{HeavisideTheta[-2 + t]} + \]
\[4 \ e^{-2t} (-1 + 2 e^t - e^4 \text{HeavisideTheta[-2 + t]} + e^{2t} \text{HeavisideTheta[-2 + t]}) + \]
\[e^{-2t} (2 e^5 + 2 e^{2t} \text{DiracDelta[-2 + t]} + e^{2t} \text{HeavisideTheta[-2 + t]} - \]
\[e^4 \text{DiracDelta'}[-2 + t] + e^{2t} \text{DiracDelta'}[-2 + t])\]

\textbf{Simplify[%]}\]
\[-4 \ e^{-2t} (1 + e^4 \text{HeavisideTheta[-2 + t]} + \]
\[e^{-2t} (e^4 \text{DiracDelta[-2 + t]} + e^t (2 + e^4 \text{HeavisideTheta[-2 + t]})\)\]
Since \( \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} \),

\[
\gamma(s) = \frac{1}{s+1} + \frac{1}{s+2} + e^{-2s} \left( \frac{1}{s+1} - \frac{1}{s+2} \right) \quad \text{and} \quad \gamma(t) = e^{-t} + e^{-2t} + (e^{-t} - e^{-2t}) h(t-2)
\]

\[
Y_2(s) = \frac{1 + e^{-2s}}{s+1} \quad \text{and} \quad Y_2(t) = e^{-t} + e^{-2(t-2)} h(t-2).
\]

\[
\therefore \gamma(t) = \begin{bmatrix} e^{-t} + e^{-2t} + (e^{-t} - e^{-2t}) h(t-2) \\ e^{-t} + e^{-2(t-2)} h(t-2) \end{bmatrix}
\]

The Mathematica solution and plot on the next page confirms the expected behavior.

**Impulsive Response:**

As an example, consider the following mass-spring-dashpot system excited from rest at \( t=0 \) by an impulsive applied force,

\[
m\gamma''(t) + y'(t) + ky(t) = S(t), \quad y(0) = 0, \quad y'(0) = 0.
\]

Applying the Laplace transform, with \( Y(s) = \mathcal{L}\{y(t)\} \) we obtain:

\[
m(s^2Y(s) - sY(0) - 0) + Y(sy(0) - 0) + kY(s) = 1
\]

\[
\therefore Y(s) = \frac{1}{ms^2 + ys + k}
\]

Recall that the response of this system to any other excitation \( f(t) \) would be:

\[
Y(s) = \left( \frac{1}{ms^2 + ys + k} \right) F(s) \quad \text{where} \quad F(s) = \mathcal{L}\{f(t)\}
\]

Therefore, the Laplace transform of the impulsive response is the system transfer function.

**Example:** Consider the linear system \( \gamma'(t) = A\gamma(t) + g(t) \), where \( A \) is a 2x2 constant matrix. The system transfer function was found to be \( (sI-A)^{-1} \). If we excite the system from rest at \( t=0 \) by \( g(t) = \begin{bmatrix} \delta(t) \\ 0 \end{bmatrix} \), the resulting output in the frequency domain,
math 4564  Systm with Impulsive Input

System: \[
\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \delta(t-2) \end{pmatrix}, \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]

a) Laplace transform solution

InverseLaplaceTransform[
(\frac{2}{(s+2)+(1+\text{Exp}[-2*s])}\text{Exp}[-2*(s+2)], \frac{(s+1)(s+2)}{(s+1)}, s, t)\]
\[\{e^{-2t} \{1+e^t + (-e^{2t}+e^{2t}) \text{HeavisideTheta}[-2+t]\}, e^{-t} \{1+e^{2} \text{HeavisideTheta}[-2+t]\}\}
\]

b) Check using DSolve

DSolve[\{y_1'[t] = -2*y_1[t] + y_2[t],
  y_2'[t] = -y_2[t] + \text{DiracDelta}[t-2], y_1[0] = 2, y_2[0] = 1\}, \{y_1[t], y_2[t]\}, t] \]
\[\{\{y_1[t] \rightarrow e^{-2t} (1+e^t - e^t \text{HeavisideTheta}[-2+t] + e^{2t} \text{HeavisideTheta}[-2+t])\}, \]
\[y_2[t] \rightarrow e^{-t} (1+e^{2} \text{HeavisideTheta}[-2+t])\}\]

c) Plot of solution components

Plot[\{e^{-2t} (1+e^t - e^t \text{HeavisideTheta}[-2+t] + e^{2t} \text{HeavisideTheta}[-2+t]),
  e^{-t} (1+e^{2} \text{HeavisideTheta}[-2+t])\}, \{t, 0, 5\}, \text{PlotRange} \rightarrow \{0, 2\}\]

Note that \(y_1(t)\) is continuous with a derivative discontinuous at \(t=2\) while \(y_2(t)\) has a jump discontinuity at \(t=2\).
\[ Y(s) = (sI - A)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \] is actually the first column of \((sI - A)^{-1}\). If we then repeat the process, exciting the system from rest at \(t = 0\) with input \( g(t) = \begin{bmatrix} 0 \\ \delta(t) \end{bmatrix} \), the resulting output \( Y(t) = (sI - A)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) yields the second column of \((sI - A)^{-1}\). The 2x2 matrix formed by stacking the two outputs then give us the system transfer function, i.e.

\[ \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = (sI - A)^{-1} \]

**Exercises:**

1. Find all possible solutions of the nonlinear integral equation:

\[ \int_0^t y(t - \lambda) \, d\lambda = \frac{t^2}{140}, \quad t \geq 0 \]

2. Solve:

\[ \gamma(t) + \int_0^t h(t - \lambda) \, d\lambda = e^{-t}, \quad t \geq 0. \]

3. If \( \gamma(t) = (f_1, f_2, f_3)(t) = 3 - 3e^{-t} - t e^{-t} - 2t + \frac{t^2}{2} \) and \( f_1(t) = f_2(t) = e^{-t} \), determine \( f_3(t), \quad 0 \leq t < \infty \).

4. a) Solve the initial value problem:

\[ y'(t) + y(t) = t + \delta(t - 1), \quad y(0) = \delta_0, \quad 0 \leq t < \infty \]

Do you anticipate that the solution \( y(t) \) will be continuous at \( t = 1 \) or not? Briefly explain.

b) Plot the solution \( y(t) \) on the interval \( 0 \leq t \leq 5 \) using Mathematica.

5. a) Solve the initial value problem:

\[ \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ \delta(t-1) \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

b) Plot both components of solution \( y(t) \) on the same graph on the interval \( 0 \leq t \leq 5 \) using Mathematica.
Application to Probability Theory:

We will discuss some applications of the Laplace Transform in probability theory.

Background:

In the presence of uncertainties, dependent variables are often modeled as random variables, quantities that take on values with corresponding probabilities.

Let $T(t)$ be a random variable taking on values on $[0,\infty)$. One might imagine $T(t)$ as the time it takes to complete a task in $t$ units of time.

Let $p(t)$ be the corresponding probability density function, i.e.,

$$p(t) \, dt = \text{probability that } T \text{ takes on values between } t \text{ and } t + dt.$$

Note: $p(t) \geq 0$ for all $0 \leq t < \infty$ and $\int_0^\infty p(t) \, dt = 1$.

Moments:

The moments of density function $p(t)$ are defined to be:

$$\mu_n = \int_0^\infty t^n p(t) \, dt, \quad n = 1, 2, 3, \ldots$$

is called the $n$th moment of $T$ (provided the integral exists.)

(Note: $\mu_1$ and $\mu_2$ can be viewed as the center of mass and moment of inertia about the origin of density function $p$,)

In particular:

a) $\mu_1 = \int_0^\infty t \, p(t) \, dt$ is called the mean or expected value of $T$, and

b) $\mu_2 = \int_0^\infty (t - \mu_1)^2 p(t) \, dt = \mu_2 - \mu_1^2$ is the variance of $T$, a centered second moment.

Density Function Examples:

Two common density functions are:

a) Uniform distribution:

$$p(t) = \begin{cases} \frac{1}{b-a}, & \alpha \leq t \leq b \\ 0, & \text{otherwise} \end{cases}$$

b) Normal distribution (Gaussian function)
b) **Exponential distribution:**

\[
p(t) = \frac{1}{\alpha} e^{-\frac{t}{\alpha}}, \quad \alpha > 0.
\]

**Note:**

\[
\mu_1 = \int_0^\infty t \cdot \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \, dt = \alpha \int_0^\infty u \cdot e^{-u} \, du = \alpha.
\]

Therefore, the constant \(\alpha\) is the mean or expected value of \(T\).

One application of the Laplace Transform arises in the formation of the characteristic function.

**Characteristic (Moment Generating) Function:**

The characteristic function is defined to be:

\[
\Phi(s) = \int_0^\infty p(t) e^{-st} \, dt = \mathcal{L}(p(t)),
\]

the Laplace transform of \(p(t)\).

Assuming that we can interchange the integration and differentiation with respect to \(s\):

\[
\frac{d^n \Phi(s)}{ds^n} = \int_0^\infty p(t) (-t)^n e^{-st} \, dt = (-1)^n \int_0^\infty p(t) t^n e^{-st} \, dt
\]

Therefore:

\[
\left. \frac{d^n \Phi(s)}{ds^n} \right|_{s=0} = (-1)^n \int_0^\infty p(t) t^n dt = (-1)^n \mu_n, \quad \text{assuming the integral exists.}
\]

and:

\[
\mu_n = (-1)^n \left. \frac{d^n \Phi(s)}{ds^n} \right|_{s=0}, \quad n = 1, 2, ...
\]

We can generate the moments in this way.

**Example:** Compute the characteristic function for the exponential distribution and use it to compute the mean and variance.

\[
\Phi(s) = \int_0^\infty \frac{1}{\alpha} e^{-\frac{t}{\alpha}} e^{-st} \, dt = \frac{1}{\alpha} \cdot \frac{1}{s + \frac{1}{\alpha}}
\]

\[
\frac{d \Phi(s)}{ds} = -\frac{1}{\alpha} \cdot \frac{1}{(s + \frac{1}{\alpha})^2} \quad \text{and} \quad \frac{d^2 \Phi(s)}{ds^2} = \frac{2}{\alpha} \cdot \frac{1}{(s + \frac{1}{\alpha})^3}
\]
\[ \mu_1 = -\frac{d\Phi(s)}{ds} \bigg|_{s=0} = 1, \quad a^2 = \alpha \quad \text{and} \quad \mu_2 = \frac{d^2\Phi(s)}{ds^2} \bigg|_{s=0} = 2, \quad a^3 = 2a^2. \]

\[ \therefore \quad \alpha^2 = \mu_2 - \mu_1^2 = 2a^3 - a^2 = a^2. \]

**Sums of Independent Random Variables:**

A second application of the Laplace Transform to probability theory arises in dealing with sums of independent random variables. As an example, suppose that \( T_1(t) \) and \( T_2(t) \) are the times needed to complete the two stages of a task. Because of uncertainties, they are modeled as random variables. Assume further that they are considered to be independent random variables; loosely speaking, the two times have no influence upon each other.

The total time needed to complete the task is therefore the random variable:

\[ T(t) = T_1(t) + T_2(t) \]

**Question:** If \( P_T(t) \) and \( P_{T_1}(t) \) are probability density functions characterizing \( T, T_1 \) and \( T_2(t) \), what is the probability density function \( P_T(t) \) characterizing the sum \( T(t) = T_1(t) + T_2(t) \)?

On can show that:

\[ P_T(t) = \int_0^\infty P_T(t ; \lambda) P_{T_1}(\lambda) d\lambda = (P_T * P_{T_1})(t) \]

the convolution of the two constituent density functions.

Therefore:

\[ P_T(t) = \mathcal{L}^{-1}(\Phi_T(s) = \mathcal{L}(P_T(t), i = 1, 2, \ldots) \]

are the constituent characteristic functions.

More generally, if \( T_1(t), T_2(t), \ldots, T_n(t) \) are a set of independent random variables and \( T(t) = T_1(t) + T_2(t) + \cdots + T_n(t) \),

\[ P_T(t) = \mathcal{L}^{-1}(\Phi_T(s) = \mathcal{L}(P_T(t), i = 1, \ldots, n) \]

**Example:** A manufacturing process involves 3 stages. The times spent to complete the stages forms a set of three independent exponentially-distributed random variables, each having an expected value of 4 hours.

a) What is the probability density function for the total manufacturing time?
b) What is the probability that the total manufacturing time 
\( T = T_1 + T_2 + T_3 \) exceeds 20 hours?

c) Use Mathematica to plot \( p(t) \) and \( P(t) \) together on the same graph.

On a separate graph plot the cumulative distribution function:
\[
F_T(r) = \int_0^r \frac{1}{4} e^{-\frac{t}{4}} dt = \Pr\{ T \leq r \}
\]
(Note: In part b) we are asked to determine \( 1 - F_T(20) \).

a) We have \( p_i(t) = \frac{1}{4} e^{-\frac{t}{4}}, \ i=1,2,3 \) where \( t \) is measured in hours, since \( \alpha_i = \mu_i = 4 \) hours.

\[
\therefore \Phi_T(s) = \mathcal{L}\left(\frac{1}{4} e^{-\frac{t}{4}}\right) = \frac{1}{4} \frac{1}{s+\frac{1}{4}}
\]

\[
\therefore \mathcal{L}^{-1}(P_T(s)) = \left(\frac{1}{4(s+\frac{1}{4})}\right)^3 = \frac{1}{64} \cdot \frac{1}{(s+\frac{1}{4})^3}
\]

Using the 1st shifting theorem:
\[
P_T(t) = \frac{1}{64} \mathcal{L}^{-1}\left(\frac{1}{(s+\frac{1}{4})^3}\right) = \frac{1}{128} \mathcal{L}^{-1}\left(\frac{2}{(s+\frac{1}{4})^3}\right) = \frac{1}{128} t^2 e^{-\frac{t}{4}}
\]

b) \( \Pr\{ T \geq 20 \} = \int_{20}^{\infty} \frac{1}{128} t^2 e^{-\frac{t}{4}} dt \approx 0.125
\]

where the integral was evaluated using Mathematica (see next page).

c) See next page for the plots.
math 4564  Manufacturing time example

b):

\[
\text{Integrate}\left[(t^2/128)*\exp(-t/4), (t, 0, \text{Infinity})\right]
\]

Out[9]= \[
\frac{37}{2 e^{5}}
\]

In[10]= \[
\text{N}\left[\frac{37}{2 e^{5}}\right]
\]

Out[10]= \[
0.124652
\]

c):

In[11]= \[
pT1[t_] := (1/4)*\exp(-t/4);
\]

In[12]= \[
pT[t_] := (t^2/128)*\exp(-t/4);
\]

In[13]= \[
\text{Plot}[\{pT1[t], pT[t]\}, \{t, 0, 40\}, \text{PlotRange} \rightarrow \{0, 0.3\}]
\]

Cumulative distribution function:

In[14]= \[
\text{Integrate}[pT[x], \{x, 0, t\}]
\]

Out[14]= \[
\frac{1}{128} \left(128 - 4 e^{-t/4} (32 + t (8 + t))\right)
\]

In[15]= \[
FT[t_] := \frac{1}{128} \left(128 - 4 e^{-t/4} (32 + t (8 + t))\right);
\]
In[16]:= Plot[FT[t], {t, 0, 50}]
Partial Differential Equations and Fourier Series:

We will solve problems modeling heat flow, wave propagation and potential theory, involving partial differential equations, using a solution technique known as Separation of Variables. The theory of Fourier Series will provide a framework for understanding the Separation of Variables solutions that we will obtain.

**Def:** Let \( u = u(x,t) \) be a dependent variable depending upon independent variables \((x,t)\) (typically space and time). The general second order linear partial differential equation in two independent variables \((x,t)\) is:

\[
a u_{xx}(x,t) + b u_{xt}(x,t) + c u_{tt}(x,t) + e u(x,t) + f u(x,t) = g
\]

where subscripts are used to denote partial derivatives; e.g. \( u_x = \frac{\partial u}{\partial x} \), and we assume \( u_{xx} = u_{tt} \).

In general the coefficients \( a, b, c, \ldots, f \) and also \( g \) can depend upon \((x,t)\) although we will consider only the constant coefficient case.

If \( g = 0 \), the equation is called **homogeneous**. If \( g \neq 0 \) the equation is called **nonhomogeneous**.

We now mention the main types of partial differential equations of mathematical physics.

**Main Types of Partial Differential Equations (PDEs) in two independent variables.**

i) **Heat Equation** (spatial variable \( x \) and time \( t \))

\[
u_t(x,t) = u_{xx}(x,t)
\]

(This type of equation is referred to as a **parabolic equation**; \( t = x^2 \) is the equation of a parabola.)

In problems modeling one-dimensional heat flow, \( u(x,t) \) will represent the temperature at position \( x \) and time \( t \).

ii) **Wave Equation** (spatial variable \( x \) and time \( t \))

\[
u_{xx}(x,t) - u_{tt}(x,t) = 0
\]

(This type of equation is referred to as a **hyperbolic equation**; \( x^2 - t^2 = 1 \) is the equation of a hyperbola.)

One might think of \( u(x,t) \) as the displacement of a plucked string from its equilibrium position at point \( x \) and time \( t \).