Two Shifting Theorems:

We derive two transform relations known as shifting theorems.

1. If \( \mathcal{L}(f(t)) = F(s) \), what is \( \mathcal{L}(e^{\alpha t} f(t)) \)?

Using the definition,

\[
\mathcal{L}(e^{\alpha t} f(t)) = \int_0^\infty e^{\alpha t} f(t) e^{-st} dt = \int_0^\infty f(t) e^{-(s-\alpha)t} dt = F(s-\alpha).
\]

\[
\therefore \mathcal{L}(e^{\alpha t} f(t)) = F(s-\alpha), \text{ where } F(s) = \mathcal{L}(f(t)).
\]

2. If \( \mathcal{L}(f(t)) = F(s) \), what is \( \mathcal{L}(f(t-\alpha)h(t-\alpha)) \), where \( 0 < \alpha < \infty \) and \( h(t) \) is the Heaviside Step Function?

A sketch of an \( f(t) \) and \( f(t-\alpha)h(t-\alpha) \) is shown below.

\[
\begin{align*}
\text{f(t)} & \quad \text{f(t-\alpha)h(t-\alpha)} \\
\end{align*}
\]

Using the definition:

\[
\mathcal{L}(f(t-\alpha)h(t-\alpha)) = \int_0^\infty f(t-\alpha)h(t-\alpha)e^{-st} dt = \int_0^\infty f(t-\alpha)e^{-(s-\alpha)t} dt
\]

Change of variables: Let \( \tau = t-\alpha \). Then \( t = \tau + \alpha \), \( dt = d\tau \) and

\[
\int_0^\infty f(\tau)e^{-st} d\tau = \int_0^\infty f(\tau)e^{-s(\tau+\alpha)} d\tau = \int_0^\infty f(\tau)e^{-s\tau}e^{-s\alpha} d\tau = e^{-s\alpha}\int_0^\infty f(\tau)e^{-s\tau} d\tau = e^{-s\alpha}F(s).
\]

\[
\therefore \mathcal{L}(f(t-\alpha)h(t-\alpha)) = e^{-s\alpha}F(s), \text{ where } 0 < \alpha < \infty \text{ and } F(s) = \mathcal{L}(f(t)).
\]

Remarks:

(i) The Shifting Theorems will often be used in the determination of inverse transforms, i.e.,

\[
\mathcal{L}^{-1}(F(s-\alpha)) = e^{\alpha t}f(t), \text{ where } f(t) = \mathcal{L}^{-1}(F(s))
\]

\[
\mathcal{L}^{-1}(e^{-s\alpha}F(s)) = f(t-\alpha)h(t-\alpha), \text{ where } f(t) = \mathcal{L}^{-1}(F(s)).
\]

(ii) The two shifting theorems exhibit an imperfect duality in that multiplication by an exponential in one domain shifts the argument.
In the other domain.

We now use the table of transform pairs, together with the shifting theorems, to compute Laplace transforms and inverse transforms.

**Examples:**

1. \( f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & 2 \leq t < \infty \end{cases} \)

   \[ f(t) = h(t-1) - h(t-2) \]

   \[ F(s) = \mathcal{L}(h(t-1)) - \mathcal{L}(h(t-2)) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} = \frac{e^{-s} - e^{-2s}}{s} \]

2. \( f(t) = \begin{cases} t^2, & 0 \leq t < 1 \\ 0, & 1 \leq t < \infty \end{cases} \)

   \[ f(t) = t^2 h(t-1) = t^2 - t^2 h(t-1) \]

   Rewrite \( t^2 h(t-1) \) as \( (t-1)^2 + 2(t-1) \) \( h(t-1) = ((t-1)^2 + 2(t-1) + 1) \) \( h(t-1) \)

   \[ F(s) = \mathcal{L}(t^2) - \mathcal{L}((t-1)^2 h(t-1)) = 2 \mathcal{L}((t-1) h(t-1)) = \mathcal{L}(h(t-1)) \]

   \[ = \frac{2}{s^3} - \frac{2e^{-s}}{s^3} - 2 \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s^3} = \frac{2}{s^3} - \frac{e^{-s}(s^2 + 2s + 2)}{s^3} \]

3. \( f(t) = \frac{e^{-t}}{2} \sin t \cos t = \frac{1}{2} e^{-t} \sin 2t \). Using the 4th shifting theorem:

   \[ F(s) = \frac{1}{2} \mathcal{L}(\sin 2t) \bigg|_{s \to s+1} = \frac{1}{2} \cdot \frac{2}{(s+1)^2 + 4} = \frac{1}{s^2 + 2s + 5} \]

4. Compute \( f(t) = \mathcal{L}^{-1}(F(s)) \) if \( F(s) = \frac{s}{s^2 + 4s + 5} \)

We solve the problem using the following steps.

(i) Noting the 2nd shifting theorem, we can obtain \( f(t) \) by computing the inverse transform of \( s/(s^2 + 4s + 5) \), shifting the argument from \( t \) to \( t-1 \) and multiplying by \( h(t-1) \).

(ii) Note that:

   \[ \frac{s}{s^2 + 4s + 5} = \frac{s}{(s+2)^2 + 1} \]

   \[ = \frac{s+2}{(s+2)^2 + 1} - 2 \cdot \frac{1}{(s+2)^2 + 1} \]
Therefore: \[ \frac{s}{s^2 + 4s + 5} = \left[ \frac{s}{s^2 + 1} - \frac{4}{s^2 + 1} \right] s \rightarrow s + 2 \]

Using the 1st shifting theorem:
\[ \mathcal{L}^{-1} \left( \frac{s}{s^2 + 4s + 5} \right) = e^{2t} (\cos 2t - 2 \sin 2t) \]

Lastly we account for the \( e^s \) multiplier using the 2nd shifting theorem to obtain:
\[ f(t) = e^{2(t-1)} (\cos (t-1) - 2 \sin (t-1)) h(t-1) \]

The next page uses Mathematica to evaluate these transforms and inverse transforms.

Note that the software often presents answers in terms of complex exponentials.
Also, recall that the cosine is an even function while the sine is an odd function, so that:
\[ \cos (1-t) = \cos (t-1), \quad \sin (1-t) = -\sin (t-1) \]

**Exercises:** Solve the following and check your answers using Mathematica.

1. Compute \( \mathcal{L}(f(t)) \) if:
   
   a) \[ f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t \leq 3 \\ 2, & 3 < t < \infty \end{cases} \]
   
   b) \[ f(t) = \begin{cases} e^t, & 0 \leq t < 2 \\ 0, & 2 \leq t < \infty \end{cases} \]
   
   c) \[ f(t) = 2 \sin 2t \sin t \]
   
   d) \[ f(t) = t^2 e^{2t} \]

2. Compute \( \mathcal{L}^{-1}(F(s)) \) if:
   
   a) \[ F(s) = \frac{e^s}{s^3} \]
   
   b) \[ F(s) = \frac{2s+1}{s^2+9} \]
   
   c) \[ F(s) = \frac{s}{(s-1)^2} \]
   
   d) \[ F(s) = \frac{e^{-2s}s}{(s-1)^2} \]

Having developed some background, we will now begin to see how the Laplace Transform can be used to solve problems.
Checking example solutions using Mathematica

1. \( f(t) = h(t-1) - h(t-2) \)
   \[
   \text{LaplaceTransform}[\text{UnitStep}[t-1] - \text{UnitStep}[t-2], t, s] = \frac{e^{-2s} - e^{-s}}{s} + \frac{e^{-s}}{s}
   \]

2. \( f(t) = t^2 (1 - h(t-1)) \)
   \[
   \text{LaplaceTransform}[t^2 (1 - \text{UnitStep}[t-1]), t, s] = \frac{2e^{-s}(2 + 2s + s^2)}{s^3} - \frac{e^{-s}}{s^3}
   \]

3. \( f(t) = e^{-t} \sin(t) \cos(t) \)
   \[
   \text{LaplaceTransform}[\text{Exp}[-t] \ast \text{Sin}[t] \ast \text{Cos}[t], t, s] = \frac{1}{5 + 2s + s^2}
   \]

4. \( F(s) = e^{-s} \left( \frac{s}{s^2 + 4s + 5} \right) \)
   \[
   \text{InverseLaplaceTransform}[\text{Exp}[-s] \ast (s/(s^2 + 4s + 5)), s, t] = \frac{1}{2} e^{-(-2it)} (-1 + i) (1 - 2i + (1 + 2i) e^{2i(-1 + t)}) \text{HeavisideTheta}[-1 + t]
   \]
   \[
   \text{FullSimplify} \left[ \frac{1}{2} e^{-(-2it)} (-1 + i) (1 - 2i + (1 + 2i) e^{2i(-1 + t)}) \text{HeavisideTheta}[-1 + t] \right] = e^{2-2t} \text{HeavisideTheta}[-1 + t] (\text{Cos}[1 - t] + 2 \text{Sin}[1 - t])
   \]
The Laplace Transform of Derivatives and Antiderivatives:

Assume that $y(t): [0, \infty) \rightarrow \mathbb{R}$ is differentiable and that $\mathcal{L}(y(t)) = Y(s)$.

What is $\mathcal{L}(y'(t))$? Starting with the definition and integrating by parts:

$$\mathcal{L}(y'(t)) = \int_0^\infty y'(t)e^{-st}dt = e^{-st}y(t)\bigg|_0^\infty + s\int_0^\infty y(t)e^{-st}dt = -y(0) + sy(s)$$

$$u = e^{-st}, \quad du = -se^{-st}dt$$
$$dv = y'(t)dt, \quad v = y(t)$$

Remark: We are assuming that the improper integrals defining $\mathcal{L}(y(t))$ and $\mathcal{L}(y'(t))$ both exist. Since $\int_0^\infty y(t)e^{-st}dt$ exists, it must be that $\lim_{t \to \infty} e^{-st}y(t) = 0$. Therefore, we obtain:

1) $\mathcal{L}(y'(t)) = sY(s) - y(0)$, where $Y(s) = \mathcal{L}(y(t))$.

b) Continuing, $\mathcal{L}(y''(t)) = \mathcal{L}((y'(t))' = s\mathcal{L}(y'(t)) - y'(0)$

$= s(sY(s) - y(0)) - y'(0)$. Therefore:

$$\mathcal{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

c) What about $\mathcal{L}\left(\int_0^t y(\lambda)d\lambda\right)$? Again starting with the definition and integrating by parts:

$$\mathcal{L}\left(\int_0^t y(\lambda)d\lambda\right) = \int_0^\infty \left(\int_0^t y(\lambda)d\lambda\right)e^{-st}dt = -\int_0^t \int_0^\infty y(\lambda)e^{-st}d\lambda e^{st}dt$$

$$u = \int_0^\infty y(\lambda)d\lambda, \quad du = y(t)dt$$

Noting that $-\left(\int_0^\infty y(\lambda)d\lambda\right)e^{st}\bigg|_0^\infty = 0$, we obtain:

$$\mathcal{L}\left(\int_0^t y(\lambda)d\lambda\right) = \frac{1}{s}Y(s), \text{ where } Y(s) = \mathcal{L}(y(t))$$

Remark: Note that, roughly speaking, differentiation in the time domain becomes multiplication by $s$ in the frequency (transform) domain while anti-differentiation becomes division by $s$. 
Application to Solving Initial Value Problems:

We begin to solve problems using the Laplace Transform by considering the initial value problems introduced at the outset.

Example: \( y(t) + y(t) = e^{-t} + 1 \), \( y(0) = 3 \).

Take Laplace Transforms of both sides of the equation, using the linearity of the transform operator,

\[
\mathcal{L}(y' (t)) + \mathcal{L}(y(t)) = \mathcal{L}(e^{-t}) + \mathcal{L}(1).
\]

Let \( Y(s) = \mathcal{L}(y(t)) \).

\[
\therefore \quad sY(s) - 3 + Y(s) = \frac{1}{s+1} + \frac{1}{s} \quad \text{and} \quad Y(s) = \frac{3}{s} + \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s(s+1)}
\]

Note:
(i) The differential equation in the time domain has been transformed into a simpler algebraic equation.
(ii) The initial condition \( y(0) = 3 \) has become part of the algebraic equation.

Inverting the transform: Using the linearity of the inverse transform operator:

\[
y(t) = 3 \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right)
\]

(i) \( \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) \): From the tables: \( \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t} \)

(ii) \( \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) \): We know \( \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t \). Therefore, using the 1st shifting theorem: \( \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = t \cdot e^{-t} \)

(iii) \( \mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right) \): We use a partial fraction decomposition (to be discussed in greater detail later.)

Set \( \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1) + Bs}{s(s+1)} = \frac{(A+B)s + A}{s(s+1)} \) and solve for the constants \( A, B \) by equating numerators. \( (A+B)s + A = 1 \)

\[
A = 1, \quad A + B = 0 \quad \Rightarrow B = -1.
\]

Alternately, \( \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \) \( \Rightarrow \frac{1}{s} = A + \frac{Bs}{s+1} \). Set \( s = 0 \) to obtain \( A = 1 \).

And:

\[
\frac{1}{s} = \frac{A(s+1) + B}{s} \quad \text{Set} \quad s = -1 \quad \text{to obtain} \quad B = -1.
\]
Therefore: \( \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \) \text{ and } \( \mathcal{L}^{-1}\left( \frac{1}{s(s+1)} \right) = 1 - e^{-t} \)

Combining results:
\[
y(t) = 3e^{-t} + te^{-t} + 1 - e^{-t} = (t+2)e^{-t} + 1.
\]

Example: Solve \( y''(t) + 3y'(t) + 2y(t) = 0 \), \( y(0) = -1 \), \( y'(0) = 4 \)
\[
\mathcal{L}(y''(t)) + 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) = 0 \quad \text{Let } Y(s) = \mathcal{L}(y(t)).
\]
\[
s^2Y(s) - s(-1) - 4 + 3(sY(s) + 1) + 2Y(s) = 0
\]
\[
(s^2 + 3s + 2)Y(s) = -s + 1 \quad \Rightarrow \quad Y(s) = \frac{-s + 1}{s^2 + 3s + 2} = \frac{-s + 1}{(s + 1)(s + 2)}
\]
Anticipating results from the theory of partial fractions:
\[
\frac{-s + 1}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2} \quad \Rightarrow \quad -s + 1 = A(s + 2) + B(s + 1). \quad \text{Set } s = -1 \text{ to obtain } A = 2.
\]
Similarly, \( -s + 1 = A(s + 2) + B. \) \text{ Set } s = -2 \text{ to obtain } B = -3.
\[
\therefore \quad Y(s) = \frac{2}{s + 1} - \frac{3}{s + 2} \quad \text{and} \quad y(t) = 2e^{-t} - 3e^{-2t}
\]
Example: Solve \( y''(t) - 2y'(t) + y(t) = 3 \), \( y(0) = 1 \), \( y'(0) = -1 \)
\[
\mathcal{L}(y''(t)) - 2\mathcal{L}(y'(t)) + \mathcal{L}(y(t)) = 3
\]
\[
\quad \text{Let } Y(s) = \mathcal{L}(y(t)). \text{ Then:}
\]
\[
s^2Y(s) - s(-1) - (sY(s) - 1) + Y(s) = 3s
\]
\[
\therefore \quad (s^2 - 2s + 1)Y(s) = s + 3 \quad \Rightarrow \quad Y(s) = \frac{s^2 - 3s + 3}{s(s - 1)^2}
\]
The appropriate partial fraction expansion is:
\[
\frac{s^2 - 3s + 3}{s(s - 1)^2} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{(s - 1)^2}
\]
We can evaluate A and C as:
\[
A = \frac{s^2 - 3s + 3}{(s - 1)^2} \bigg|_{s = 0} = 3 \quad ; \quad C = \frac{s^2 - 3s + 3}{s} \bigg|_{s = 1} = 1
\]
Since \( \frac{s^2 - 3s + 3}{s(s - 1)^2} = \frac{3}{s} + \frac{B}{s - 1} + \frac{1}{(s - 1)^2} \) we can set \( s = 2 \), for example and obtain \( B = -2 \).
Therefore: \( y(s) = \frac{3s - 2}{s - 1} + \frac{1}{(s-1)^2} \) and \( y(t) = 3 - 2e^{t} + t e^{t} = 3 + (t-2)e^{t} \)

We now summarize the Method of Partial Fractions.

**Method of Partial Fractions:**

The method of partial fractions applies to a rational function, i.e. a ratio of polynomials \( \frac{N(s)}{D(s)} \), in which the degree of the numerator \( N(s) \) is strictly less than the degree of the denominator polynomial \( D(s) \).

The form of the expansion is determined by the roots or factors of the denominator \( D(s) \). The next page, replicating Table 5.2 of the text, summarizes the proper form of these expansions.

Note: A quadratic \( s^2 + 2ax + b \) has roots \( s = -a \pm \sqrt{4a^2 - 4b} \)
or \( s = -a \pm \sqrt{a^2 - b} \). If \( a < \sqrt{b} \), the roots are complex and the quadratic is referred to as irreducible.

**Examples:** Using the table we determine the proper form of the partial fraction expansion (without evaluating the constants).

1. \( \frac{s^2 + s + 2}{s^5 + 4s^4 + 3s^3} = \frac{s^2 + s + 2}{s^3(s^2 + 4s + 3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} + \frac{E}{s+3} \)
2. \( \frac{s^2 + s + 2}{(s+1)(s^2 + 4s + 5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+5} \)
   2 irreducible quadratics
3. \( \frac{s^2 + s + 2}{s^4 - 16} = \frac{s^2 + s + 2}{(s^2 - 4)(s^2 + 4)} = \frac{s^2 + s + 2}{(s-2)(s+2)(s^2 + 4)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{Cs+D}{s^2 + 4} \)
### TABLE 5.2

<table>
<thead>
<tr>
<th>Denominator Polynomial Factors and Their Corresponding Terms in the Partial Fraction Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Denominator Factor</strong></td>
</tr>
<tr>
<td>1. Simple real root $s - \alpha$</td>
</tr>
<tr>
<td>2. Repeated real root $(s - \alpha)^n$</td>
</tr>
<tr>
<td>3. Irreducible quadratic factor $s^2 + \omega^2$</td>
</tr>
<tr>
<td>4. Repeated irreducible quadratic factor $(s^2 + \omega^2)^n$</td>
</tr>
<tr>
<td>5. Irreducible quadratic factor $s^2 + 2\alpha s + \beta^2, \beta^2 &gt; \alpha^2$</td>
</tr>
<tr>
<td>6. Repeated irreducible quadratic factor $(s^2 + 2\alpha s + \beta^2)^n, \beta^2 &gt; \alpha^2$</td>
</tr>
</tbody>
</table>

Find $L^{-1}\{F(s)\}$, where

$$F(s) = \frac{s^2 + 4}{s^4 - s^2}.$$

**Solution:** The function $F(s)$ is a rational function in which the degree of numerator $N(s) = s^2 + 4$ is less than that of denominator $D(s) = s^4 - s^2$. The denominator factors into

$$D(s) = s^2(s^2 - 1) = s^2(s - 1)(s + 1).$$

Therefore, the denominator has $s = 0$ as a repeated real root and $s = \pm 1$ as
Example: Solve \( y''(t) - 2y'(t) + y(t) = e^{-t} + \sin(t) + t \), \( y(0) = 3, y'(0) = 4 \).

Let \( Y(s) = \mathcal{L}\{y(t)\} \). Then:

\[
S^2 Y(s) - sY(s) - 1 - 2(sY(s) - 3) + Y(s) = \frac{1}{S+1} + \frac{1}{S^2+1} + \frac{1}{S^2}
\]

\[
(s^2 - 2s + 1)Y(s) = 3s^2 + 1 + \frac{1}{S+1} + \frac{1}{S^2+1} + \frac{1}{S^2}
\]

\[
Y(s) = \frac{3s^2 - 5}{(s-1)^2} + \frac{1}{(s-1)(s+1)} + \frac{1}{(s-1)^2(s+1)} + \frac{1}{s^2(s-1)^2}
\]

(i) \( \frac{3s^2 - 5}{(s-1)^2} = A_1 + \frac{A_2}{s-1} \)

(ii) \( \frac{1}{(s-1)(s+1)} = \frac{B_1}{s-1} + \frac{B_2}{s+1} \)

(iii) \( \frac{1}{(s-1)^2(s+1)} = \frac{C_1}{s-1} + \frac{C_2}{s+1} + \frac{C_3 s + C_4}{s^2 + 1} \)

(iv) \( \frac{1}{s^2(s-1)^2} = \frac{D_1 s + D_2}{s} + \frac{D_3 s + D_4}{s-1} + \frac{D_5}{(s-1)^2} \)

The partial fraction calculations become tedious. We will use Mathematica to complete the solution. We can:

(a) Use the Inverse Laplace Transform command to obtain \( y(t) \) directly,

or,

(b) Use the Apart command to generate the partial fraction expansion and then use tables to obtain \( y(t) \).

We obtain: \( y(t) = \frac{e^{-t}}{4} + \frac{e^t}{4} + t + 2 + \frac{\cos(t)}{2} \)

Exercises:

1. Compute \( \mathcal{L}\left( \int_1^t x^4 \, dx \right) \), using the fact that \( \int_1^t x^4 \, dx = \int_0^t x^4 \, dx - \int_0^1 x^4 \, dx \)

   and \( \int_0^1 x^4 \, dx = \frac{1}{5} \).

2. Compute \( \mathcal{L}\left( \int_0^t \int_0^\sigma x \sin(2\sigma) \, d\sigma d\lambda \right) \) without performing any integrations.

3. Compute \( \mathcal{L}\left( \int_0^t f(\lambda) \, d\lambda \right) \) where \( f(t) \) is shown graphically.
Solution of initial value problem

\[ y''(t) - 2y'(t) + y(t) = e^{-t} + \sin(t) + t, \quad y(0) = 3, \quad y'(0) = 1 \]

\[
y(s) = \left( \frac{1}{(s-1)^2} \right) \left( 3s - 5 + \frac{1}{s+1} + \frac{1}{s^2+1} + \frac{1}{s^2} \right)
\]

1. Use of the InverseLaplaceTransform command:

\[
\text{InverseLaplaceTransform} [\left( \frac{1}{((s-1)^2)} * (3s - 5 + \frac{1}{s+1} + \frac{1}{s^2+1} + \frac{1}{s^2}), s, t \right]
\]

\[
2 + \frac{e^{-t}}{4} + \frac{e^t}{4} + t + \frac{\cos[t]}{2}
\]

2. Use of the Apart command:

\[
\text{Apart}[\left( \frac{1}{((s-1)^2)} * (3s - 5 + \frac{1}{s+1} + \frac{1}{s^2+1} + \frac{1}{s^2}) \right]
\]

\[
\frac{1}{4(-1+s)} + \frac{1}{s^2} + \frac{2}{s} + \frac{1}{4(1+s)} + \frac{s}{2(1+s^2)}
\]

Using the tables:

\[
y(t) = \frac{e^{-t}}{4} + t + 2 + \frac{e^t}{4} + \frac{\cos[t]}{2}
\]
Exercises (cont.):

4. Solve the initial value problem \( y'(t) + 2y(t) = e^{-2t} \), \( y(0) = -2 \) using Laplace transforms.

5. Solve the initial value problem \( y''(t) + 4y'(t) = 0 \), \( y(0) = 1 \), \( y'(0) = 4 \) using Laplace transforms.

6. The initial value problem \( y''(t) + \alpha y'(t) + \beta y(t) = 0 \), \( y(0) = y_0 \), \( y'(0) = y'_0 \)
has transform domain solution \( Y(s) = \frac{2s+5}{s^2+6s+10} \). Determine the constants \( \alpha, \beta, y_0 \) and \( y'_0 \).

7. Compute \( \mathcal{L}^{-1} \left( \frac{-2s+5}{s^2+6s+10} \right) \). Check your answer using Mathematica's Inverse Laplace Transform command. (Use the FullSimplify command or other appropriate command to obtain an answer expressed in terms of real-valued functions.)

8. Develop the partial fraction expansion for the given functions.
   You need not evaluate the constants.

   a) \( F(s) = \frac{s^3 + 2s - 1}{s^4 + 5s^2 + 4} \)

   b) \( F(s) = \frac{s}{s^3 - 1} \) (Hint: Use synthetic division to show that \( s^3 - 1 = (s-1)(s^2+s+1) \).)

   c) \( F(s) = \frac{2s - 7}{s(s^2 - 9)^2} \)

   d) \( F(s) = \frac{2s - 7}{s^5 - 2s^4 + s^3} \)

   e) \( F(s) = \frac{2s - 7}{(s^5 - 3s^2 + 2)^2} \)

9. Derive a formula for \( \mathcal{L}(y''(t)) \) in terms of \( Y(s) = \mathcal{L}(y(t)) \), assuming that \( y(t) \) is sufficiently differentiable and that all relevant transforms (i.e. \( \mathcal{L}(y(t)), \mathcal{L}(y'(t)), \mathcal{L}(y''(t)), \mathcal{L}(y'''(t)) \)) exist.
We now consider an initial value problem whose solutions via transforms is arguably more direct.

Example: Solve: \( y'(t) + 2y(t) = f(t) \), \( y(0) = 0 \) where

\[
 f(t) = \begin{cases} 
 0, & 0 \leq t < 1 \\
 1, & 1 \leq t < 2 \\
 0, & 2 \leq t < \infty 
\end{cases}
\]

a) Time domain (direct) solution:

Consider the problem on each time subinterval.

(i) \( 0 \leq t < 1 \): \( y'(t) + 2y(t) = 0 \), \( y(0) = 0 \). The general solution is \( y(t) = Ce^{-2t} \). Setting \( y(0) = C = 0 \) leads to \( y(t) = 0 \), \( 0 \leq t < 1 \).

(ii) \( 1 \leq t < 2 \): \( y'(t) + 2y(t) = 1 \). To obtain the initial condition at \( t = 1 \), we argue that the most ill-behaved or singular term on the left side of the differential equation is \( y'(t) \) and it undergoes a jump discontinuity at \( t = 1 \). Therefore \( y(t) \) must be continuous at \( t = 1 \) and we set \( y(1) = 0 \).

\[
 y'(t) + 2y(t) = 1, \quad y(1) = 0. \quad \therefore (Ce^{2t})' = e^t \quad \text{and} \quad e^{2t} y(t) = \frac{1}{2} e^{2t} + C
\]

\[
 y(t) = \frac{1}{2} + Ce^{2t}. \quad \text{Setting} \quad t = 1: \quad y(1) = \frac{1}{2} + Ce^{-2} = 0 \quad \Rightarrow \quad C = -\frac{1}{2} e^2
\]

\[
 \text{and} \quad y(t) = \frac{1}{2} - \frac{1}{2} e^{2(t-1)}, \quad 1 \leq t < 2.
\]

(iii) \( 2 \leq t < \infty \): The equation again becomes \( y'(t) + 2y(t) = 0 \). We require solution \( y(t) \) to be continuous at \( t = 2 \). Therefore we solve

\[
 y'(t) + 2y(t) = 0, \quad y(2) = \frac{1}{2} - \frac{1}{2} e^2, \quad 2 \leq t < \infty
\]

\[
 y(t) = Ce^{-2t}. \quad \text{Imposing the initial condition:}
\]

\[
 y(2) = Ce^{-4} = \frac{1}{2} - \frac{1}{2} e^2 \quad \Rightarrow \quad C = \frac{1}{2} e^4 - \frac{1}{2} e^2
\]

\[
 y(t) = \frac{1}{2} e^{-2(t-2)} - \frac{1}{2} e^{-2(t-1)}, \quad 2 \leq t < \infty
\]

**Summarizing:** \( y(t) = \begin{cases} 
 0, & 0 \leq t < 1 \\
 \frac{1}{2} - \frac{1}{2} e^{2(t-1)}, & 1 \leq t < 2 \\
 \frac{1}{2} e^{-2(t-2)} - \frac{1}{2} e^{-2(t-1)}, & 2 \leq t < \infty
\end{cases} \)

An alternate time-domain solution can be obtained using the integrating factor.