Four Special Cases:

As a prelude to solving the general Dirichlet problem for the rectangle, we list the Separation of Variables solutions for the four special cases where nonzero boundary data exists on only one edge of the boundary.

1. \[ u_{1}(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{\pi n x}{a}\right) \sinh\left(\frac{\pi n y}{b}\right) \]
   \[ a_n = \frac{2}{b} \int_{0}^{b} f_1(y) \sinh\left(\frac{\pi n y}{b}\right) dy \]
   \[ n = 1, 2, 3, \ldots \]

2. \[ u_{2}(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{\pi n x}{b}\right) \sin\left(\frac{\pi n y}{a}\right) \]
   \[ b_n = \frac{2a}{\pi n} \int_{0}^{a} f_2(x) \sin\left(\frac{\pi n x}{a}\right) dx \]
   \[ n = 1, 2, 3, \ldots \]

3. \[ u_{3}(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{\pi n (a-x)}{b}\right) \sinh\left(\frac{\pi n y}{b}\right) \]
   \[ c_n = \frac{2b}{\pi n} \int_{0}^{b} f_3(y) \sinh\left(\frac{\pi n y}{b}\right) dy \]
   \[ n = 1, 2, 3, \ldots \]

4. \[ u_{4}(x,y) = \sum_{n=1}^{\infty} d_n \sinh\left(\frac{\pi n (b-y)}{a}\right) \sin\left(\frac{\pi n x}{a}\right) \]
   \[ d_n = \frac{2a}{\pi n} \int_{0}^{a} f_4(x) \sin\left(\frac{\pi n x}{a}\right) dx \]
   \[ n = 1, 2, 3, \ldots \]

Remarks:

i) The solutions have trigonometric (sin) dependence in the variable having zero boundary conditions on two parallel sides (x=0 \& x=a or y=0 \& y=b) and hyperbolic (sinh) dependence in the variable having zero boundary data on only one side.

ii) The argument of sinh(\cdot) must vanish on the one side having zero boundary data, e.g. sinh(\pi n x/a) when u(x,0)=0 and sinh(\pi n (a-x)/b) when u(x,a)=0.
The General Solution:

Since Laplace's equation is linear and homogeneous, a linear combination of solutions will be a solution. In particular, a sum of the four special solutions formed previously will be a solution of the following Dirichlet problem:

\[
\begin{align*}
\Delta U(x, y) &= 0, \quad 0 < x < a, \\
U(a, y) &= f_1(y), \quad 0 < y < b, \\
U(x, b) &= f_2(x), \quad 0 < x < a, \\
U(0, y) &= f_3(y), \quad 0 < y < b, \\
U(x, 0) &= f_4(x), \quad 0 < x < a.
\end{align*}
\]

\[
U(x, y) = U_1(x, y) + U_2(x, y) + U_3(x, y) + U_4(x, y).
\]

This is not the general solution, however, because the special cases considered assume the boundary data values at the four vertices to be zero, i.e.

\[f_1(b) = f_2(a) = f_3(0) = f_4(0) = 0\]

The general Dirichlet problem will typically have continuous boundary data that need not vanish, however, at the four vertices.

General Solution:

Given the Dirichlet problem:

\[
\begin{align*}
\Delta U(x, y) &= 0, \quad 0 < x < a, 0 < y < b, \\
U(0, y) &= \phi_1(y), \quad 0 < y < b, \\
U(x, b) &= \phi_2(x), \quad 0 < x < a, \\
U(0, y) &= \phi_3(y), \quad 0 < y < b, \\
U(x, 0) &= \phi_4(x), \quad 0 < x < a.
\end{align*}
\]

assume that the boundary data has the following vertex values:

\[\phi_1(a) = \phi_2(0) = \phi_1, \quad \phi_2(b) = \phi_2(a) = \phi_2, \quad \phi_3(0) = \phi_3(b) = \phi_3, \quad \phi_4(a) = \phi_4(0) = \phi_4,\]

We will construct a solution as follows;
(i) Find a solution of Laplace's equation, call it \( U(x,y) \), that interpolates the boundary vertex values, i.e.
\[
U(x,0) = \psi_1, \quad U(x,b) = \psi_2, \quad U(0,b) = \psi_3, \quad U(0,0) = \psi_4
\]
(ii) Form \( U(x,y) = u(x,y) - \psi(x,y) \). The problem for \( U(x,y) \) becomes:

\[
\begin{align*}
\Delta U(x,y) &= 0, \quad 0 < x < a, \quad 0 < y < b \\
U(x,0) &= g_1(y) - \psi(x,0) \\
U(x,b) &= g_2(x) - \psi(x,b) \\
U(0,y) &= g_3(y) - \psi(0,y) \\
U(0,0) &= g_4(x) - \psi(0,0)
\end{align*}
\]

\( \Delta U(x,y) = 0 \), \( 0 < x < a, \quad 0 < y < b \)

\( U(x,0) = g_1(y) - \psi(x,0) = \frac{f_1(y)}{b}, \quad 0 \leq y \leq b \)

\( U(x,b) = g_2(x) - \psi(x,b) = \frac{f_2(x)}{a}, \quad 0 \leq x \leq a \)

\( U(0,y) = g_3(y) - \psi(0,y) = \frac{f_3(y)}{b}, \quad 0 \leq y \leq b \)

\( U(0,0) = g_4(x) - \psi(0,0) = \frac{f_4(x)}{a}, \quad 0 \leq x \leq a \)

Note that the boundary data values at the four vertices will be zero. Therefore, we can solve for \( U(x,y) \) as the sum of the four special cases.

The general solution will therefore be:

\[
u(x,y) = U(x,y) + \psi(x,y) = \sum_{i=1}^{4} u_i(x,y) + \psi(x,y)
\]

Construction of \( \psi(x,y) \):

We need a solution of Laplace's equation satisfying:

\[
u(x,0) = \psi_1, \quad \nu(x,b) = \psi_2, \quad \nu(0,b) = \psi_3, \quad \nu(0,0) = \psi_4
\]

where \( \psi_i, i=1,\ldots,4 \) are four (generally arbitrary) constants.

Let:

\[
\psi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy
\]

where \( \alpha_1,\ldots,\alpha_4 \) are constants. Note that:

\[
\Delta \psi(x,y) = \psi_{xx}(x,y) + \psi_{yy}(x,y) = 0
\]
We now show that we can always determine values of the \( \{ \alpha_i \}_{i=1}^4 \) to satisfy the vertex constraints,

\[
\begin{align*}
\mathcal{U}(0,0) &= \alpha_1 = \mathcal{U}_4 \\
\mathcal{U}(0,0) &= \alpha_1 + \alpha_2 = \mathcal{U}_4 + \alpha \mathcal{U}_2 = \mathcal{U}_1 \quad \therefore \quad \alpha_2 = \frac{\mathcal{U}_1 - \mathcal{U}_4}{\alpha} \\
\mathcal{U}(0,b) &= \alpha_1 + b \alpha_3 = \mathcal{U}_4 + b \alpha_3 = \mathcal{U}_3 \quad \therefore \quad \alpha_3 = \frac{\mathcal{U}_3 - \mathcal{U}_4}{b} \\
\mathcal{U}(a,b) &= \alpha_1 + a \alpha_2 + b \alpha_3 + ab \alpha_4 = \mathcal{U}_4 + a \left( \frac{\mathcal{U}_1 - \mathcal{U}_4}{\alpha} \right) + b \left( \frac{\mathcal{U}_3 - \mathcal{U}_4}{b} \right) + ab \alpha_4 = \mathcal{U}_2 \\
\therefore \quad \alpha_4 &= \frac{\mathcal{U}_1 - \mathcal{U}_2 + \mathcal{U}_3 - \mathcal{U}_4}{ab}
\end{align*}
\]

**Example:** Solve the following Dirichlet problem:

\[
\begin{align*}
\Delta u(x,y) &= 0, \quad 0 < x < 2, \quad 0 < y < 1 \\
u(x,0) &= x = \mathcal{U}_4(x), \quad 0 \leq x \leq 2 \\
u(x,1) &= 2 - x = \mathcal{U}_3(x), \quad 0 \leq x \leq 2 \\
u(0,y) &= 2 \sin(\pi y/2) = \mathcal{U}_2(y), \quad 0 \leq y \leq 1 \\
u(2,y) &= 2 \cos(\pi y/2) = \mathcal{U}_1(y), \quad 0 \leq y \leq 1
\end{align*}
\]

1. Determine \( \mathcal{U}(x,y) \):

The vertex values of the boundary data are:

\[
\begin{align*}
\mathcal{U}_4(2) &= \mathcal{U}_1(0) = \mathcal{U}_1 = 2 \\
\mathcal{U}_2(0) &= \mathcal{U}_2(1) = \mathcal{U}_2 = 0 \\
\mathcal{U}_3(0) &= \mathcal{U}_3(1) = \mathcal{U}_3 = 2 \\
\mathcal{U}_4(0) &= \mathcal{U}_4(1) = \mathcal{U}_4 = 0
\end{align*}
\]

\[
u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy
\]

\[
\begin{align*}
\mathcal{U}(x,0) &= \alpha_1 = 0 \\
\mathcal{U}(2,y) &= \alpha_1 + 2 \alpha_2 = 2 \alpha_2 = 2 \quad \therefore \quad \alpha_2 = 1 \\
\mathcal{U}(0,1) &= \alpha_1 + \alpha_3 = \alpha_3 = 2 \\
\mathcal{U}(2,1) &= \alpha_1 + 2 \alpha_2 + \alpha_3 + 2 \alpha_4 = \alpha_4 + 2 \text{ at } 2 \\
\therefore \quad \alpha_4 &= -2
\end{align*}
\]

\[
u(x,y) = x + 2y - 2xy
\]

2. Form the Dirichlet problem for \( \mathcal{U}(x,y) = u(x,y) - \mathcal{U}(x,y) \):
\[ U(x, y) = 2 \sin \left( \frac{\pi y}{2} \right) - 2y \]

\[ U(x, y) = 0, \quad 0 < x < 2, \quad 0 < y < 1 \]
\[ U(2, y) = \frac{\partial}{\partial x} U(2, y) = 2 \cos \left( \frac{\pi y}{2} \right) + 2y - 2, \quad 0 < y < 1 \]
\[ U(x, 1) = \frac{\partial}{\partial y} U(x, 1) = 0, \quad 0 < x < 2 \]
\[ U(0, y) = \frac{\partial}{\partial x} U(0, y) = 2 \sin \left( \frac{\pi y}{2} \right) - 2y, \quad 0 < y < 1 \]
\[ U(x, 0) = \frac{\partial}{\partial y} U(x, 0) = 0, \quad 0 < x < 2. \]

**Note:** Two of the special solutions, \(u_2\) and \(u_4\), are solutions of the Dirichlet problem with zero boundary values on all four rectangle sides. The unique solutions are \(U_2(x, y) = U_4(x, y) = 0\). (See next page.)

Therefore:

\[ U(x, y) = U(x, y) + v(x, y) = U_1(x, y) + U_3(x, y) + v(x, y). \]

The solution is determined and plotted using Mathematica on the following pages.

**Dirichlet Problem for the Circle:**

Introduce polar coordinates:

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ r^2 = x^2 + y^2 \]
\[ \tan \theta = \frac{y}{x} \]

**Problem:**

\[ \Delta U(r, \theta) = 0, \quad 0 < r < \rho, \quad 0 < \theta < 2\pi \]
\[ U(\rho, \theta) = f(\theta), \quad 0 < \theta < 2\pi \]

In polar coordinates:

\[ \Delta U(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U(r, \theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U(r, \theta)}{\partial \theta^2} \]

We require:

(i) boundary condition \(f(\theta)\) to be \(2\pi\)-periodic, i.e.
\[ f(\theta + 2\pi) = f(\theta) \]

(ii) solution \(U(r, \theta)\) to be \(2\pi\)-periodic and bounded within the domain.
Solution of Dirichlet Problem (Notes, p. 105)  W. Kohler

Problem:
\[ U_{xx}(x, y) + U_{yy}(x, y) = 0, \quad 0 < x < 2, \quad 0 < y < 1 \]
\[ U(2, y) = 2 \cos \left( \frac{\pi y}{2} \right) + 2y - 2, \quad 0 \leq y \leq 1 \]
\[ U(x, 1) = 0, \quad 0 \leq x \leq 2 \]
\[ U(0, y) = 2 \sin \left( \frac{\pi y}{2} \right) - 2y, \quad 0 \leq y \leq 1 \]
\[ U(x, 0) = 0, \quad 0 \leq x \leq 2 \]

The desired solution is:
\[ u(x,y) = U(x,y) + v(x,y) \text{ where } v(x,y) = x + 2y - 2xy. \]

Note that:
(i) The boundary data for \( U(x,y) \) has zero values at the rectangle vertices. That is the purpose of the \( v(x,y) \) function.
(ii) \( U(x,y) \) will generally be the sum of the 4 special case solutions \( U(x,y) = \sum_{n=1}^{4} u_n(x,y) \). In this case, however, two of them will be zero.

Therefore the desired solution will have the form: \( u(x,y) = u_1(x,y) + u_3(x,y) + v(x,y) \).

\[ u_1 \text{ problem:} \]
\[ \Delta u_1(x, y) = 0, \quad 0 < x < 2, \quad 0 < y < 1 \]
\[ u_1(2, y) = 2 \cos \left( \frac{\pi y}{2} \right) + 2y - 2, \quad 0 \leq y \leq 1 \]
\[ u_1 = 0 \text{ on other three rectangle sides.} \]

\[ u_3 \text{ problem:} \]
\[ \Delta u_3(x, y) = 0, \quad 0 < x < 2, \quad 0 < y < 1 \]
\[ u_3(0, y) = 2 \sin \left( \frac{\pi y}{2} \right) - 2y, \quad 0 \leq y \leq 1 \]
\[ u_3 = 0 \text{ on other three rectangle sides.} \]

Solving the \( u_1 \) problem:
\[ u_1(x, y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi x) \sin(n\pi y) \]
\[ a_n \sinh(2n\pi) = 2 \int_{0}^{1} \left( 2 \cos \left( \frac{\pi y}{2} \right) + 2y - 2 \right) \sin(n\pi y) \, dy, \quad n = 1, 2, 3, \ldots \]

Integrate \( (2 \cos(n\pi y/2) + 2y - 2) \sin(n\pi y), \{y, 0, 1\} \)

\[ 2\pi \sin(n\pi) \]

\[ 2n\pi + (-2 - 4n^2)(-2 + \pi) \sin(n\pi) \]

\[ \frac{n^2(-1 + 4n^2)\pi}{n^2 - (-1 + 4n^2)\pi} \]

Since \( n \) is a positive integer, \( \sin(n\pi) = 0 \).
\begin{align*}
\text{Solving the } u_3 \text{ problem:} \\
u_3(x, y) &= \sum_{n=1}^{\infty} c_n \sinh (n\pi (2-x)) \sin (n\pi y) \\
c_n \sinh (2n\pi) &= 2 \int_0^1 \left(2 \sin \left(\frac{n\pi y}{2}\right) - 2y\right) \sin (n\pi y) \, dy, \quad n = 1, 2, 3, \ldots \\
\text{Integrate } &\left(2 \sin \left(\frac{n\pi y}{2}\right) - 2y\right) \sin (n\pi y), \quad \{y, 0, 1\} \\
\frac{2 \left(\frac{n\pi \cos(n\pi)}{1-4n^2} - \sin(n\pi)\right)}{n^2 \pi^2} \\
\text{Since } n \text{ is a positive integer, } \sin(n\pi) = 0 \text{ and } \cos(n\pi) = (-1)^n.
\end{align*}
In[52] = \[ u3[x_, y_, N_] := \text{Sum[c[n] \ast Sinh[n \ast Pi \ast (2 - x)] \ast Sin[n \ast Pi \ast y], \{n, 1, N\}];} \]

Check:

In[54] = \text{Plot3D[u3[x, y, 100], \{x, 0, 2\}, \{y, 0, 1\}, \text{PlotRange} \rightarrow \{0, 0.5\}]} 

Out[54]=

In[60] = \text{Plot[\{2 \ast Sin[Pi \ast y / 2] - 2 \ast y, u3[0, y, 100]\}, \{y, 0, 1\}]} 

Out[60]=

Solving the Problem:

In[67] = \[ v[x_, y_] := x + 2 \ast y - 2 \ast x \ast y; \]

In[68] = \[ u[x_, y_, N_] := u1[x, y, N] + u3[x, y, N] + v[x, y]; \]
\textbf{In[00] = } \texttt{Plot3D[u[x, y, 100], \{x, 0, 2\}, \{y, 0, 1\}]}

\textbf{Out[00]}
The Dirichlet Problem has a unique solution.

This assertion states that there can be at most one solution. The common argument assumes there are two solutions and shows they must be equal.

We present an argument for the 3-dimensional problem using the familiar form of the Divergence Theorem.

**Problem:** \( \Delta u(x) = 0, \quad x \in \Omega \)
\( u = f \) on \( \partial \Omega \)

Assume that \( u_1(x) \) and \( u_2(x) \) are two solutions. Form the difference \( w(x) = u_1(x) - u_2(x) \).

Then:
\[
\Delta w(x) = \Delta (u_1(x) - u_2(x)) = \Delta u_1(x) - \Delta u_2(x) = 0, \quad x \in \Omega
\]
\( w = f - f = 0 \) on \( \partial \Omega \).

Consider:
\[
\iiint_{\Omega} \nabla \cdot (w \nabla w) \, dV
\]

On the one hand:
\[
\iiint_{\Omega} \nabla \cdot (w \nabla w) \, dV = \iiint_{\Omega} (\nabla w \cdot \nabla w + w \Delta \nabla \, dV = \iiint_{\Omega} |\nabla w|^2 \, dV
\]

On the other hand, using the Divergence Theorem:
\[
\iiint_{\Omega} \nabla \cdot (w \nabla w) \, dV = \iint_{\partial \Omega} w \nabla w \cdot \mathbf{n} \, d\Sigma = 0 \text{ since } w = 0 \text{ on } \partial \Omega.
\]

\( \therefore \) \( \iiint_{\Omega} |\nabla w|^2 \, dV = 0 \Rightarrow \nabla w(x) = 0 \text{ in } \Omega, \therefore w(x) \text{ is constant in } \Omega. \)

However, \( w(x) \) is continuous on \( \overline{\Omega} = \Omega \cup \partial \Omega \) and \( w = 0 \) on \( \partial \Omega \).

\( \therefore \) \( w(x) = u_1(x) - u_2(x) = 0 \text{ in } \overline{\Omega} \text{ or } u_1(x) = u_2(x). \)

**Remark:** For the Neumann problem, if \( \nabla u \cdot \mathbf{n} = \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \text{ on } \partial \Omega \),
then \( \nabla w \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \) and one can again conclude that \( w(x) = 0 \text{ constant} \).

In this case \( u_1(x) \) and \( u_2(x) \) can differ by an additive constant.
Separation of Variables Solution:

We look for solutions of Laplace's equation, $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0$, of the form $u(r, \theta) = R(r) \Theta(\theta)$, requiring these solutions to be 2\pi-periodic and bounded in the domain.

Substituting:

$$R'' \Theta + \frac{R'}{r} \Theta + \frac{1}{r^2} R \Theta'' = 0 \Rightarrow \frac{r^2 R'' + r R'}{r} = -\Theta'' = \sigma$$

:. $r^2 R''(r) + r R'(r) - \sigma R(r) = 0$, $0 < r < \rho$

$\Theta'' + \sigma \Theta = 0$, $0 \leq \theta \leq 2\pi$

We assume (without loss of generality) that $\sigma$ is real-valued.

We consider $\Theta'' + \sigma \Theta = 0$ and look for nontrivial 2\pi-periodic solutions.

Three possibilities:

i) $\sigma = 0$: $\Theta''(\theta) = 0 \Rightarrow \Theta(\theta) = k_1 \theta + k_2$. 2\pi-periodicity requires $k_1 = 0$ but $k_2$ is not constrained. \(. \Rightarrow \Theta(\theta) = 1\) is a 2\pi-periodic solution.

Now consider: $r^2 R''(r) + r R'(r) = 0$. Let $\nu(r) = R'(r)$. Then we obtain the first order equation $r^2 \nu''(r) + r \nu'(r) = 0$ or $r \nu'(r) + \nu(r) = 0$ which we can solve and then compute an antiderivative to obtain $R(r)$.

$r \nu'(r) + \nu(r) = (r \nu(r))' = 0 \Rightarrow r \nu(r) = c_1 \Rightarrow R(r) = c_1 \ln r + c_2$.

Demanding that $R(r)$ be bounded in the domain requires that $c_1 = 0$ but $c_2$ is not constrained. Take $R_0(r) = 1$.

$. \Rightarrow$ we obtain $u(r, \theta) = R_0(r) \Theta_0(\theta) = 1$ when $\sigma = 0$.

ii) $\sigma < 0$: Let $\sigma = -\lambda^2 < 0$. Then:

$$\Theta''(\theta) - \lambda^2 \Theta(\theta) = 0 \Rightarrow \Theta(\theta) = k_1 e^{\lambda \theta} + k_2 e^{-\lambda \theta}$$

2\pi-periodicity requires $k_1 = k_2 = 0$. We obtain no nontrivial solutions in this case.
lii) \( r > 0 \): Let \( \alpha = \lambda^2 \):

\[
\Theta(\theta) + \lambda^2 \Theta(\theta) = 0 \Rightarrow \Theta(\theta) = k_1 \cos \lambda \theta + k_2 \sin \lambda \theta
\]

We obtain 2\( \pi \)-periodic solutions if \( \lambda_n = n \) (\( n = 1, 2, 3, \ldots \))

\[
\Theta_n(\theta) = a_n \cos n \theta + b_n \sin n \theta, \quad n = 1, 2, 3, \ldots
\]

where \( a_n \) and \( b_n \) are arbitrary constants. Note that since \( \cos \) and \( \sin \) are even and odd functions, respectively, we gain nothing new if \( n \) is a negative integer.

The radial equation becomes:

\[
r^2 R''(r) + r R'(r) - n^2 R(r) = 0, \quad 0 < r < p
\]

This equation is called a Cauchy-Euler (or Euler) equation. Note that the exponent of \( r \) in each term matches the number of derivatives on \( R(r) \).

Therefore, we look for solutions of the form \( R(r) = r^m \).

Substituting:

\[
r^2 u(u-1)r^{u-2} + u - n^2 r^u = (u(u-1) + u - n^2) r^u = 0, \quad 0 < r < p.
\]

\[
u(u-1) + u - n^2 = 0 \quad \text{and we obtain:}
\]

\[
R(r) = c_1 r^n + c_2 r^{-n}, \quad n = 1, 2, 3, \ldots
\]

The demand that \( R(r) \) be bounded on \( 0 < r < p \) requires that \( c_2 = 0 \).

Taking \( c_1 = 1 \), we have \( R_n(r) = r^n \) and:

\[
\Theta_n(\theta) = R_n(r) \Theta_n(\theta) = r^n \left( a_n \cos n \theta + b_n \sin n \theta \right), \quad n = 1, 2, 3, \ldots
\]

where \( a_n, b_n \) are arbitrary constants.

We form the solution template:

\[
u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n \left( a_n \cos n \theta + b_n \sin n \theta \right)
\]

and look to satisfy the boundary condition \( u(p, \theta) = f(\theta) \).

Therefore:

\[
u(p, \theta) = f(\theta) = a_0 + \sum_{n=1}^{\infty} p^n \left( a_n \cos n \theta + b_n \sin n \theta \right), \quad 0 \leq \theta \leq 2\pi
\]

Note that this a Fourier Series representation of the 2\( \pi \)-periodic function \( f(\theta) \).
Comparing, for example, \( \cos(n\theta) \) with \( \cos\left(\frac{m\pi}{x}\right) \), we see that \( \pi \) corresponds to \( \lambda \) and the coefficient formulas become:

\[
\begin{align*}
\rho^n a_n &= \frac{1}{\pi} \int_{\pi} f(\theta) \cos(n\theta) \, d\theta, \quad n = 0, 1, 2, 3, \ldots \\
\rho^n b_n &= \frac{1}{\pi} \int_{\pi} f(\theta) \sin(n\theta) \, d\theta, \quad n = 1, 2, 3, \ldots
\end{align*}
\]

**Example:** Solve the Dirichlet problem:

\[
\begin{align*}
\Delta u(r, \theta) &= 0, \quad 0 < r < 3, \quad 0 \leq \theta \leq 2\pi \\
u(3, \theta) &= 3\cos^2 \theta + 4\sin^3 \theta, \quad 0 \leq \theta \leq 2\pi \\
u(r, \theta) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n \left( a_n \cos n\theta + b_n \sin n\theta \right)
\end{align*}
\]

Imposing the boundary condition:

\[
\begin{align*}
u(3, \theta) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 3^n (a_n \cos n\theta + b_n \sin n\theta) = 3\cos^2 \theta + 4\sin^3 \theta, \quad 0 \leq \theta \leq 2\pi
\end{align*}
\]

**Note:**

\[
\begin{align*}
3\cos^2 \theta + 4\sin^3 \theta &= \frac{3}{2} + \frac{3}{2} \cos 2\theta + 2 \sin \theta \left( 1 - \cos 2\theta \right) \\
&= \frac{3}{2} + \frac{3}{2} \cos 2\theta + 2 \sin \theta - (\sin 3\theta - \sin \theta) \\
&= \frac{3}{2} + \frac{3}{2} \cos 2\theta + 3 \sin \theta - \sin 3\theta
\end{align*}
\]

Comparing, we have:

\[
\begin{align*}
\frac{\alpha_0}{2} &= \frac{3}{2}, \quad 3^2 a_2 = \frac{3}{2}, \quad a_n = 0, \quad n \neq 0, 2 \\
3 b_3 &= 3, \quad 3^3 b_3 = -1, \quad b_n = 0, \quad n \neq 1, 3
\end{align*}
\]

**Solution:**

\[
u(r, \theta) = \frac{3}{2} + \frac{3}{2} \left( \frac{\pi}{3} \right)^2 \cos 2\theta + r \sin \theta - \left( \frac{\pi}{3} \right)^3 \sin 3\theta
\]

**Example:** Solve the Dirichlet problem:

\[
\begin{align*}
\Delta u(r, \theta) &= 0, \quad 0 < r < 2, \quad 0 \leq \theta \leq 2\pi \\
u(2, \theta) &= \begin{cases} 2\cos \theta, & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\
0, & \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \text{ and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}
\end{cases}
\end{align*}
\]

If we plot the periodic extension of \( u(2, \theta) \) vs. \( \theta \) we see that
the boundary condition \( u(2, \Theta) \) is an even function of \( \Theta \). Therefore
the Fourier Series reduces to the Cosine Series:

\[
    u(2, \Theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^n a_n \cos(n \Theta),
\]

i.e. \( a_n = 0, \ n = 1, 2, 3, \ldots \) and

\[
    2^n a_n = \frac{1}{\pi} \int_{\Theta}^{\frac{\pi}{2}} 2 \cos \Theta \cos(n \Theta) d\Theta = \frac{4}{\pi} \int_{0}^{\frac{\pi}{2}} \cos \Theta \cos(n \Theta) d\Theta =
\]

\[
    = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \left( \cos((n-1)\Theta) + \cos((n+1)\Theta) \right) d\Theta.
\]

\[
    a_0 = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \cos \Theta d\Theta = \frac{4}{\pi} \sin \Theta \bigg|_{0}^{\frac{\pi}{2}} = 4.
\]

\[
    2^n a_1 = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} (1 + \cos 2\Theta) d\Theta = \frac{2}{\pi} \left( \Theta + \sin 2\Theta \right) \bigg|_{0}^{\frac{\pi}{2}} = 1, \quad \therefore \ a_1 = \frac{1}{2}
\]

\[
    2^n a_n = \frac{2}{\pi} \left( \frac{\sin \left( \frac{(n-1)\pi}{2} \right)}{n-1} + \frac{\sin \left( \frac{(n+1)\pi}{2} \right)}{n+1} \right) \bigg|_{0}^{\frac{\pi}{2}} = \frac{2}{\pi} \left( \frac{\sin \left( \frac{(n-1)\pi}{2} \right)}{n-1} + \frac{\sin \left( \frac{(n+1)\pi}{2} \right)}{n+1} \right)
\]

Note: \( \sin \left( \frac{(n\pm 1)\pi}{2} \right) = \sin \left( \frac{n\pi}{2} \right) \cos \left( \frac{\pi}{2} \right) \pm \cos \left( \frac{n\pi}{2} \right) \sin \left( \frac{\pi}{2} \right) = \pm \cos \left( \frac{n\pi}{2} \right) \)

\[
    \therefore 2^n a_n = \frac{2}{\pi} \cos \left( \frac{n\pi}{2} \right) \left( \frac{1}{n-1} + \frac{1}{n+1} \right) = -4 \cos \left( \frac{n\pi}{2} \right), \quad n \geq 2.
\]

**Solution:**

\[
    u(r, \Theta) = \frac{a_0}{2} + \left( \frac{r}{2} \right) \cos \Theta - \frac{4}{\pi} \sum_{n=2}^{\infty} \left( \frac{r}{2} \right)^n \left( \frac{\cos \left( \frac{n\pi}{2} \right)}{n(n-1)} \right) \cos (n \Theta)
\]

See Mathematica solution surface plot on next pages.

**Dirichlet Problem for the Annulus:**

**Problem:**

\[
    \Delta u(r, \Theta) = 0, \quad 0 \leq r < a, \quad 0 \leq \Theta \leq 2\pi
\]

\[
    u(r, \Theta) = f(\Theta), \quad 0 \leq \Theta \leq 2\pi
\]

\[
    u(a, \Theta) = \phi(\Theta), \quad 0 \leq \Theta \leq 2\pi
\]
Dirichlet problem for the circle

Problem:
\[ \Delta u(r, \theta) = 0, \quad 0 < r < 2, \quad 0 \leq \theta \leq 2\pi \]
\[ u(2, \theta) = f(\theta) = \begin{cases} 
2 \cos(\theta) & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\
0 & -\pi \leq \theta < -\frac{\pi}{2}, \quad \frac{\pi}{2} < \theta \leq \pi
\end{cases} \]

Solution:

\[ u(r, \theta) = \frac{8}{\pi} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)) \]
\[ a_n = \frac{2}{\pi} \int_0^{\pi/2} 2 \cos(\theta) \cos(n\theta) \, d\theta, \quad n = 1, 2, 3, \ldots \]
\[ b_n = 0, \quad n = 1, 2, 3, \ldots \text{ because } f(\theta) \text{ is an even function of } \theta. \]

Evaluation of coefficients:

\[ (2 / \pi) \times \text{Integrate}[2 \times \cos[t] \times \cos[n \times t], \{t, 0, \pi / 2\}] \]
\[ \frac{4 \cos[\frac{\pi n}{2}]}{(-1 + n^2) \pi} \]

\[ a[n_] := \frac{4 \cos[\frac{\pi n}{2}]}{(-1 + n^2) \pi}; \]

Indeterminacy removed when \( n = 1 \).

\[ \text{Limit}[-\frac{4 \cos[\frac{\pi n}{2}]}{(-1 + n^2) \pi}, \, n \to 1] \]
\[ 1 \]

\[ a[1] = 1; \]

\[ u[r, \theta, N_] := a[0] / 2 + \text{Sum}[(r / 2)^n \times a[n] \times \cos[n \times t], \{n, 1, N\}]; \]

Solution surface is plotted, using a 100 partial sum as an adequate approximation.
ParametricPlot3D[\{r \cdot \cos[t], r \cdot \sin[t], u[r, t, 100]\}, \{r, 0, 2\}, \{t, 0, 2 \cdot \pi\}]

Separation of Variables Solution:

The derivation follows that for the circle. However, the functions that were rejected because of the boundedness requirement as \( r \to 0 \) are now retained because \( 0 < r \leq r_0 \). This proves to be essential because there are now two boundary conditions to be satisfied.

The Separation of Variables solution template is:

\[
U(r, \theta) = \frac{A_0}{2} + \frac{B_0}{2} \ln r + \sum_{n=1}^{\infty} r^{-n}(A_n \cos(n \theta) + B_n \sin(n \theta))
\]

\[0 < r_0 \leq r \leq r_0, \quad 0 \leq \theta \leq 2\pi\]

Imposing the boundary conditions:

\[
U(r_0, \theta) = f(\theta) = \frac{A_0}{2} + \frac{B_0}{2} \ln r_0 + \sum_{n=1}^{\infty} \left( (A_n + B_n) \cos(n \theta) + (A_n B_n)^n \sin(n \theta) \right)
\]

\[U(r_0, \theta) = A_0 \frac{B_0}{2} \ln r_0 + \sum_{n=1}^{\infty} \left( (A_n + B_n) \cos(n \theta) + (A_n B_n)^n \sin(n \theta) \right)
\]

Let:

\[
\frac{\pi}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n \theta) d\theta = \alpha_n \quad \text{;} \quad \frac{\pi}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n \theta) d\theta = \beta_n, \quad n=0,1,2,\ldots
\]

Then, we obtain the following systems of equations:

\[
\left[ \begin{array}{c}
1 \\
1
\end{array} \right] \ln \left[ \begin{array}{c}
A_0 \\
B_0
\end{array} \right] = \left[ \begin{array}{c}
\alpha_0 \\
\beta_0
\end{array} \right] ; \quad \left[ \begin{array}{c}
A_n \\
B_n
\end{array} \right] = \left[ \begin{array}{c}
\alpha_n \\
\beta_n
\end{array} \right] ; \quad \left[ \begin{array}{c}
A_n \\
B_n
\end{array} \right] = \left[ \begin{array}{c}
\alpha_n \\
\beta_n
\end{array} \right]
\]

Note that the determinants:

\[
\left| \begin{array}{c}
1 \\
1
\end{array} \right| \ln \frac{A_0}{B_0} = \ln r - \ln r_0 = \ln (\frac{r}{r}) = 0 \quad \text{and} \quad \left| \begin{array}{c}
r_n \\
r_n
\end{array} \right| = (r_n + r_n)^n - (r_n - r_n)^n \neq 0
\]

since \( \frac{p_n}{p_n} < 1 \).

\[
\therefore \text{the systems of equations have unique solutions for the coefficients.}
\]