Homework #2 Due date: Thursday, October 11, 2005

Math 5474: Finite Difference Methods for PDEs

Name:

(1) (Exercise 2.2.3, page 57, Strikwerda)

For the equation

\[ u_t + a u_{xx} = f, \]  

(1)

consider the following finite difference scheme:

\[ v_m^{n+1} = \frac{1}{2}(v_{m+1}^n + v_{m-1}^n) - \frac{1}{2} a k h^{-3}(v_{m+2}^n - 2v_{m+1}^n - 2v_{m-1}^n + v_{m-2}^n) + k f_m^n. \]  

(2)

Show that the finite difference scheme (2) is consistent with equation (1) if \( k^{-1}h^2 \) tends to zero as \( h \) and \( k \) tend to zero.

(2) (Exercise 2.3.1, page 60, Strikwerda)

Use the unstable forward-time forward-space scheme for \( u_t + u_x = 0 \) with the initial data

\[ u_0(x) = \begin{cases} 
1 - |x| & \text{if } |x| \leq 1, \\
0 & \text{otherwise},
\end{cases} \]

on the interval \([-1, 3]\) for \( 0 \leq t \leq 1 \). Use a grid spacing of 0.1 and \( \lambda = 0.8 \). Demonstrate that the instability grows by approximately \( |g(\pi)| \) per time step. Comment on the appearance of the graph of \( v_m^n \) as a function of \( m \). Use the
boundary condition \( u(t, -1) = 0 \) at the left boundary and use \( v_M^{n+1} = v_{M-1}^{n+1} \) at the right boundary.

(3) (Exercise 2.3.2, page 60, Strikwerda)

Use the unstable forward-time central-space scheme for \( u_t + u_x = 0 \) with the following two sets of initial data on the interval \([-1, 3]\) for \( 0 \leq t \leq 1 \):

\[
(a) \quad u_0(x) = \begin{cases} 
1 - |x| & \text{if } |x| \leq 1, \\
0 & \text{otherwise,}
\end{cases}
\]

\[
(b) \quad u_0(x) = \sin(x).
\]

Use a grid spacing of 0.1 and \( \lambda = 0.8 \). Demonstrate that the instability is evident sooner with the less smooth initial data (a) than it is for the smooth data (b). Show that the growth in the instability for each case is approximately \( |g(\pi/2)| \). For (a) use the boundary condition \( u(t, -1) = 0 \), and for (b) use the boundary condition \( u(t, -1) = -\sin(1 + t) \). Use \( v_M^{n+1} = v_{M-1}^{n+1} \) at the right boundary.

(3) (Exercise 2.3.3, page 60, Strikwerda)

Solve the initial value problem for equation

\[
u_t + \left(1 + \frac{1}{4}(3 - x)(1 + x)\right) u_x = 0
\]

on the interval \([-1, 3]\) with the Lax-Friedrichs scheme with \( \lambda = 0.8 \). Demonstrate that the instability phenomena occur where \(|a(t, x)\lambda| > 1\) and where there are discontinuities in the solution. Use the initial data

\[
u_0(x) = \begin{cases} 
1 - |x| & \text{if } |x| \leq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Specify the solution to be 0 at both boundaries. Compute up to the time of 0.2 and use successively smaller values of \( h \) to show the location of the instability.