Controller Reduction for Large-scale Systems
by Krylov Projection Methods

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Consider an $n^{th}$ order plant $G(s)$ with $m$ inputs and $p$ outputs:

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \Leftrightarrow \quad G(s) = C(sI - A)^{-1}B + D$$

An $n_K^{th}$ order stabilizing controller $K(s)$:

$$K(s) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \quad \Leftrightarrow \quad K(s) = C_K(sI - A_K)^{-1}B_K + D_K$$

LQG, $\mathcal{H}_\infty$ control designs $\Rightarrow n_K = n \Rightarrow$

(i) Complex hardware (ii) Degraded accuracy (iii) Degraded computational speed

Obtain $K_r(s)$ of order $r \ll n_K$ to replace $K(s)$ in the closed loop.
Controller reduction via frequency weighting

- Small open loop error $\|K(s) - K_r(s)\|_\infty$ not enough.  \(\Rightarrow\)

- Minimize the weighted error:
  \[
  \|W_o(s)(K(s) - K_r(s))W_i(s)\|_\infty.
  \]

- How to obtain the weights $W_o(s)$ and $W_i(s)$?

- If $K(s)$ and $K_r(s)$ have the same number of unstable poles and if
  \[
  \|K(s) - K_r(s)\|G(s)[I + G(s)K(s)]^{-1}\|_\infty < 1, \text{ or}\]
  \[
  \|[I + G(s)K(s)]^{-1}G(s)[K(s) - K_r(s)]\|_\infty < 1,
  \]
  \(\Rightarrow\) $K_r(s)$ stabilizes $G(s)$. 


• For stability considerations:
\[ W_i(s) = I \quad \text{and} \quad W_o(s) = [I + G(s)K(s)]^{-1}G(s) \quad \text{or} \]
\[ W_o(s) = I \quad \text{and} \quad W_i(s) = G(s)[I + G(s)K(s)]^{-1}. \]

• To preserve closed-loop performance:
\[ W_i(s) = [I + G(s)K(s)]^{-1} \quad \text{and} \quad W_o(s) = [I + G(s)K(s)]^{-1}G(s). \]

• Solved by frequency-weighted balancing (Anderson and Liu [1989], Schelfhout and De Moor [1996], Varga and Anderson [2002]).

• Requires solving two Lyapunov equations of order \( n + n_K \).
\[
A_i\mathcal{P} + \mathcal{P}A_i^T + B_iB_i^T = 0, \quad A_o^T\mathcal{Q} + \mathcal{Q}A_o + C_o^TC_o = 0,
\]

• \( A_i, B_i: \quad K(s)W_i(s), \quad A_o, C_o: \quad W_o(s)K(s) \)

• Balance \( \mathcal{P} \) and \( \mathcal{Q} \).
Controller-reduction via Krylov Projection

- Given $2r$ interpolation points: $\{\sigma_i\}_{i=1}^r$ and $\{\mu_j\}_{j=1}^r$
- Set $V = \text{Ran} \left[ (\mu_1 I - A_K)^{-1} B_K \cdots (\mu_r I - A_K)^{-1} B_K \right]$
- And $Z = \text{Ran} \left[ (\sigma_1 I - A_K^T)^{-1} C_K^T \cdots (\sigma_r I - A_K^T)^{-1} C_K^T \right]$
- $K_r(s) := \begin{bmatrix} Z^T A_K V & Z^T B_K \\ C_K V & D_K \end{bmatrix}$ (with $Z^T V = I_r$)
- $T(s) = \text{full-order closed-loop}, \quad T_r(s) = \text{reduced-order closed-loop}$
- $K_r(s)$ interpolates $K(s)$ at $\{\sigma_i\}$ and $\{\mu_j\}$. Moreover,

\[ T_r(s) \text{ interpolates } T(s) \text{ at } \sigma_i \text{ and } \mu_j \text{ for } i, j = 1, \ldots, r \]

- Gugercin et. al. [MTNS, 2004], Van Dooren et. al. [2004]
• Achieved while staying in numerically efficient Krylov-framework.

| How to choose interpolation points $\sigma_i$ and $\mu_j$? |

• **Starting point**: $\mathcal{H}_2$ error expression (Gugercin and Antoulas [2003])

• $K(s)$ with poles $\lambda_i$ and $K_r(s)$ with poles $\tilde{\lambda}_j$

• $\phi_i := K(s)(s - \lambda_i) \big|_{s=\lambda_i}$ and $\tilde{\phi}_j := K_r(s)(s - \tilde{\lambda}_j) \big|_{s=\tilde{\lambda}_j}$.

$$
\|K(s) - K_r(s)\|^2_{\mathcal{H}_2} = \sum_{i=1}^{n} \phi_i \left( K(-\lambda_i) - K_r(-\lambda_i) \right) + \\
\sum_{j=1}^{r} \tilde{\phi}_j \left( K_r(-\tilde{\lambda}_j) - K(-\tilde{\lambda}_j) \right).
$$

• **Open loop error** due to mismatch at $-\lambda_i$ and $-\tilde{\lambda}_j$

$$
\implies \text{Arrange } \sigma_i \text{ and } \mu_j \text{ so that } \sigma_i = -\lambda_i \quad \text{and} \quad \mu_j = -\tilde{\lambda}_j
$$

• As good as/better than balancing: Gugercin and Antoulas [2003]
Optimal $\mathcal{H}_2$ approximation problem

- Find $K_r(s)$ that minimizes $\|K(s) - K_r(s)\|_{\mathcal{H}_2}$

- Meier and Luenberger [1967]
  First-order necessary conditions:

\[
K_r(s)|_{s=-\hat{\lambda}_i} = K(s)|_{s=-\hat{\lambda}_i} \quad \text{and} \quad \frac{dK_r(s)}{ds}|_{s=-\hat{\lambda}_i} = \frac{dK(s)}{ds}|_{s=-\hat{\lambda}_i}
\]

- $\hat{\lambda}_i =$ poles of $K_r(s)$

- Requires successive rational Krylov steps
An Iterative Rational Krylov Iteration: (Gugercin, Antoulas and Beattie [2004])

1. Choose $\sigma_i$ for $i = 1, \ldots, r$.

2. $V = \text{Span} \left[ (\sigma_1 I - A_K)^{-1} B_K \cdots (\sigma_r I - A_K)^{-1} B_K \right]$,

3. $Z = \text{Span} \left[ (\sigma_1 I - A_K^T)^{-1} C_K^T \cdots (\sigma_r I - A_K^T)^{-1} C_K^T \right]$, $Z^T V = I_r$.

4. while [relative change in $\sigma_j$] $> \epsilon$
   (a) $A_r = Z^T A_K V$,
   (b) $\sigma_i \leftarrow -\lambda_i(A_r)$ for $i = 1, \ldots, r$
   (c) $V = \text{Span} \left[ (\sigma_1 I - A_K)^{-1} B_K \cdots (\sigma_r I - A_K)^{-1} B_K \right]$.
   (d) $Z = \text{Span} \left[ (\sigma_1 I - A_K^T)^{-1} C_K^T \cdots (\sigma_r I - A_K^T)^{-1} C_K^T \right]$, $Z^T V = I_r$.

5. $A_r = Z^T A_K V$, $B_r = Z^T B_K$, $C_r = C_K V$

\[\Downarrow\]
Optimal $\mathcal{H}_2$ reduction via Krylov projection
• After convergence, first-order $H_2$ optimality constraints are satisfied.

• The optimal solution of a restricted $H_2$ minimization problem.

• How to modify for the controller reduction problem?

• How to reflect the weights $W_i(s)$ and $W_o(s)$, the closed loop information, in the reduction step?
• Let $W_i(s) = I$ and $W_o(s) = [I + G(s)K(s)]^{-1} G(s)$ ⇒

- $A_K \mathcal{P} + \mathcal{P} A_K^T + B_K B_K^T = 0$  
- $A_w^T \mathcal{Q} + \mathcal{Q} A_w + C_w^T C_w = 0$. 

unweighted Lyapunov eq.  

weighted Lyapunov eq.

• In the rational Krylov setting: $\Pi = Z V^T$

• $Z = \mathcal{K}(A^T, C^T, \sigma_i)$,  
  and  
  $V = \mathcal{K}(A, B, \mu_j)$

• $Z$ and $\sigma_i$: Reflect $W_o(s)$: the closed-loop information.  
  $\sigma_i = jw_i$ over the region where $W_o(jw)$ is dominant

• $V$ and $\mu_j$: Obtain in an (optimal) open loop sense.  
  $\mu_j$: From an iterative rational Krylov iteration
An Iterative Rational Krylov Iteration for Controller Reduction:

1. Choose $\sigma_i = jw_i$, for $i = 1, \ldots, r$ where $w_i$ is chosen to reflect $W_o(jw)$.
2. $Z = \text{Span } [(\sigma_1 I - A_{T_K}^{-1} C_{T_K}^T \cdots (\sigma_r I - A_{T_K}^{-1} C_{T_K}^T )]$ with $Z^T Z = I_r$.
3. $V = Z$
4. while [relative change in $\mu_j] > \epsilon$
   (a) $A_r = Z^T A_K V$
   (b) $\mu_j \leftarrow -\lambda_i(A_r)$ for $j = 1, \ldots, r$
   (c) $V = \text{Span } [(\mu_1 I - A_K)^{-1} B_K \cdots (\mu_r I - A_K)^{-1} B_K ]$ with $Z^T V = I_r$.
5. $A_r = Z^T A_K V$, $B_r = Z^T B_K$, $C_r = C_K V$

$Z \Rightarrow K_r(s)$ includes the closed – loop information

$V \Rightarrow K_r(s)$ is optimal in a restricted $\mathcal{H}_2$ sense

\[ \Pi = ZV^T \]
International Space Station Module 1R:

- \( n = 270 \). \( G(s) \) is lightly damped \( \Rightarrow \) Long-lasting oscillations.
- \( K(s) \) is designed to remove these oscillations. \( n_K = 270 \).

- Reduce the order to \( r = 19 \) using iterative Rational Krylov and to \( r = 23 \) using one-sided frequency weighted balancing
• **FWBR**: Frequency-weighted balancing with $W_i(s) = I$ and $W_o(s) = [I + G(s)K(s)]^{-1} G(s)$.

• **IRK-CL**: Iterative Rational Krylov - Closed Loop version: $\sigma_i$ reflect the weight $W_o(s)$.

\[ Bode Plot of W_o(s) = T(s) = [I + G(s)K(s)]^{-1} G(s) \]

• $\sigma_i = j \cdot \logspace(-1, 2, 10)$ rad/sec
ISS Example: Bode Plots of reduced closed-loop systems

Relative Errors

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{H}_\infty$ error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T - T_{FW20}$</td>
<td>$3.88 \times 10^1$</td>
</tr>
<tr>
<td>$T - T_{FW23}$</td>
<td>$5.63 \times 10^{-1}$</td>
</tr>
<tr>
<td>$T - T_{IRK-CL}$</td>
<td>$1.47 \times 10^{-1}$</td>
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Relative Errors

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<th>$\mathcal{H}_2$ error</th>
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<tbody>
<tr>
<td>$T - T_{FW20}$</td>
<td>$3.90 \times 10^0$</td>
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<tr>
<td>$T - T_{FW23}$</td>
<td>$1.88 \times 10^{-1}$</td>
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<tr>
<td>$T - T_{IRK-CL}$</td>
<td>$3.57 \times 10^{-2}$</td>
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Weighted Errors

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<tr>
<td>$W_i(K - K_{FW20})$</td>
<td>$0.984 &lt; 1$</td>
</tr>
<tr>
<td>$W_i(K - K_{FW23})$</td>
<td>$0.416 &lt; 1$</td>
</tr>
<tr>
<td>$W_i(K - K_{IRK-CL})$</td>
<td>$0.365 &lt; 1$</td>
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### Relative Errors

<table>
<thead>
<tr>
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<th>$\mathcal{H}_\infty$ Error</th>
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<tbody>
<tr>
<td>$K - K_{FW20}$</td>
<td>$1.15 \times 10^0$</td>
</tr>
<tr>
<td>$K - K_{FW23}$</td>
<td>$9.85 \times 10^{-1}$</td>
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<tr>
<td>$K - K_{IRK-CL}$</td>
<td>$6.55 \times 10^{-1}$</td>
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<tr>
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<th>$\mathcal{H}_2$ Error</th>
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<tbody>
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<td>$K - K_{FW20}$</td>
<td>$9.96 \times 10^{-1}$</td>
</tr>
<tr>
<td>$K - K_{FW23}$</td>
<td>$9.71 \times 10^{-1}$</td>
</tr>
<tr>
<td>$K - K_{IRK-CL}$</td>
<td>$1.08 \times 10^{-2}$</td>
</tr>
</tbody>
</table>
Evolution of the closed-loop $\mathcal{H}_\infty$ error

Evolution of the closed-loop $\mathcal{H}_2$ error

Evolution of the unstable Pole

Evolution of the parameters throughout the iteration
An Unstable Model:

- $n=2000$. $K(s)$ of order $n_K = 2000$ stabilizes the model.

- $K(s)$ has four unstable poles.
- Reduce the order to \( r = 14 \): Stabilizing controller
- \( K_r(s) \) has 4 unstable poles as desired.
Conclusions and Future Work:

- Rational Krylov method for controller reduction
  - Guaranteed closed-loop matching while staying in Krylov framework
  - No Lyapunov equations need to be solved

- How to select the interpolation points?

- Iterative rational Krylov Algorithm: closed-loop version
  - $\sigma_i$ reflect the weights (closed-loop information)
  - $\mu_j$ lead to an optimal open loop controller
  - Combination of open and closed loop reduction
  - Leads to small $\mathcal{H}_2$ and $\mathcal{H}_\infty$ closed-loop errors and stabilizes for low-order controllers

- Optimality of the resulting controller in the closed-loop sense?

- How to extend to an iterative controller-plant reduction method?