On the Velocity Projection for Polyhedral Skorokhod Problems

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Abstract
We consider the Skorokhod problem with oblique reflection on a convex polyhedral domain. A collection of complementarity problems is identified which describe the velocity projection map appearing in the absolutely continuous formulation of the Skorokhod problem. Under Assumption 2.1 of Dupuis and Ishii, insuring the Lipschitz continuity of the Skorokhod map, we show that the solvability of the complementarity problems is equivalent to their Assumption 3.1 on the existence of the discrete projection map.

1 Introduction
The Sokorkhod Problem is a natural mechanism for modeling dynamics of processes confined to a set $G \subseteq \mathbb{R}^n$. In general the idea is to specify a set $d(x)$ of admissible directions of reflection to be used in restoring the state to $G$ when its dynamics would push it out through $x \in \partial G$. To be more specific, given a function $\psi : [0,T] \rightarrow \mathbb{R}^n$ (of some appropriate class) with $\psi(0) \in D$, the Skorokhod Problem is to find a finite measure $\ell$ on $[0,T]$, and measurable functions $\gamma(\cdot)$, $\phi(\cdot)$ so that the following hold on $[0,T]$:

• $\phi(t) = \psi(t) + \int_{[0,t]} \gamma(s) \, d\ell$
• $\phi(t) \in G$
• $\gamma(t) \in d(\phi(t))$ for $\ell$-almost all $t$
• $\ell([0,t]) = \int_{[0,t]} 1_{\partial G}(\phi(s)) \, d\ell$.

The map $\Gamma : \psi(\cdot) \mapsto \phi(\cdot)$ is called the Skorokhod map, to whatever extent it is well-defined. When $\partial G$ is smooth and $d(x)$ is a smooth vector field on $\partial G$, the Skorokhod map on continuous functions is one approach to treating reflecting diffusion processes in $G$; see [13] and [1].

We are concerned here only with the situation where $G \subseteq \mathbb{R}^n$ is a closed convex set of points $x$ defined by a collection of $N$ linear inequalities

$$x \cdot n_i \geq c_i, \quad \text{for } i = 1, \ldots, N$$

where $n_i$ are unit vectors and $c_i$ are scalars. This case is of particular interest for applications to queueing theory. See the many references in [10], and also [5] where $\Gamma$ is called the “oblique reflection mapping”. For $x \in G$,

$I(x) = \{ i : x \cdot n_i = c_i \}$

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denotes the set of indices for which the constraints are active. Note that $\partial G$ consists of those $x \in G$ for which $I(x)$ is nonempty. A “restoration vector” $d_i$ is assigned to each constraint (1), normalized by $n_i \cdot d_i = 1$. The sets $d(x)$ are defined in terms of them: for each $x \in \partial G$,
\[
d(x) = \{ \gamma = \sum_{i \in I(x)} \alpha_i d_i : \alpha_i \geq 0, |\gamma| = 1 \}.
\]

This polyhedral version of the Skorokhod problem has been carefully studied by Dupuis and Ishii [9] and Dupuis and Ramanan [10]. Our arguments below employ many ideas and constructions from those papers, as well as [4]. We follow their notation (with the one exception that we use $\Pi(x)$ rather than $\pi(x)$ for the discrete projection of (3) below).

The $n_i, d_i$ must satisfy some conditions for the Skorokhod problem to be well-posed and have decent continuity properties. Dupuis and Ishii [9] present separate sufficient conditions for continuity and existence. A general sufficient condition for continuity of the Skorokhod map is the following (we retain the same title as in [9] and [10]):

**Assumption 2.1.** There exists a compact, convex set $B \subseteq \mathbb{R}^n$ with $0 \in B^c$, such that for each $i = 1, \ldots, N$ and $z \in \partial B$, and any inward normal $v$ to $B$ at $z$,
\[
|z \cdot n_i| < 1 \quad \text{implies} \quad v \cdot d_i = 0.
\]

We refer the reader to [10] and [9] for details of this definition, properties of $B$ and techniques to verify its existence. Assumption 2.1 will be a hypothesis for all our results below. The major implication of Assumption 2.1 is Lipschitz continuity of the Skorokhod map on functions in $D([0,T], \mathbb{R}^n)$ (right continuous functions having left limits):
\[
sup_{0 \leq t \leq T} |\phi_2(t) - \phi_1(t)| \leq K \sup_{0 \leq t \leq T} |\psi_2(t) - \psi_1(t)|
\]
for any two solution pairs $\psi_i(\cdot) = \Gamma(\phi_i(\cdot))$ in $D([0,T], \mathbb{R}^n)$, see Theorem 2.2 of [9].

Assumption 2.1 does not necessarily imply the existence of a solution $\phi(\cdot)$ to the Skorokhod problem for a given $\psi(\cdot)$. (But it does imply uniqueness.) The discussion in [9] links existence of solutions to the Skorokhod Problem to existence of a discrete projection map $\Pi : \mathbb{R}^n \to G$ such that $y = \Pi(x)$ satisfies the following properties
\[
y = x \text{ if } x \in G,
y \in \partial G \text{ and } y - x \in d(y) \text{ if } x \notin G.
\]

The connection of $\Pi(x)$ with the Skorokhod problem is that, for any $x_0 \in G$, if we take
\[
\psi(t) = \begin{cases} x_0 & \text{for } t \leq 1 \\
x & \text{for } 1 < t \end{cases}
\]
\[
\phi(t) = \begin{cases} x_0 & \text{for } t \leq 1 \\
y & \text{for } 1 < t \end{cases},
\]
then $\phi(\cdot)$ solves the Skorokhod problem $\phi(\cdot) = \Gamma(\psi(\cdot))$ iff $y = \Pi(x)$, as one easily checks. Thus, under Assumption 2.1, $\Pi(x)$ is unique, if it exists. Moreover if $\Pi(x_i)$ exists for two $x_i$ values, $i = 1, 2$, then (2) implies
\[
|\Pi(x_2) - \Pi(x_1)| \leq K|x_2 - x_1|.
\]

The existence of $\Pi(x)$ is Assumption 3.1 of [9]. We do not assume that here. Rather Theorems 2 and 3 below will establish an equivalence between the existence of $\Pi(x)$ and the existence of solutions to the following family of complementarity problems: given $x \in \partial G$ and $v \in \mathbb{R}^n$, find $w \in \mathbb{R}^n$ such that
\[
w = v + \sum_{i \in I(x)} \beta_i d_i, \quad \text{where } \beta_i \text{ are scalars satisfying}
\]
\[
\beta_i \geq 0, \ n_i \cdot w \geq 0, \text{ and } \beta_i (n_i \cdot w) = 0 \text{ for each } i \in I(x).
\]

We will say that *the solution is unique* when there is at most one such $w$ (for a given $x$ and $v$), even though there might be more than one choice of $\beta$, associated with it.
Theorem 2 below will show that the solution of the complementarity problem (5) is the value \( w = \pi(x, v) \) of the velocity projection map of [9], defined by

\[
\pi(x, v) = \lim_{\Delta \to 0} \frac{\Pi(x + \Delta v) - x}{\Delta}.
\]  

(6)

Dupuis and Ishii show that, in the class of absolutely continuous \( \psi(\cdot) \), the Skorokhod problem \( \phi = \Gamma(\psi) \) can be expressed in differential form:

\[
\dot{\phi}(t) = \pi(x(t), \dot{\psi}(t)) \text{ almost surely.}
\]

The argument is contained in the proof of Theorem 5.1,[9].

The identification of the complementarity problems with \( \pi(x, v) \) is the primary motive for this work. Several recent papers ([2], [3], [6], [7], [8]) consider differential games for systems whose state dynamics involve a Skorokhod problem. Formulations of the Hamilton-Jacobi-Isaacs equations associated with these games involve \( \pi(x, v) \); see [6] in particular. This explains current interest in \( \pi(x, v) \).

2 Some Preliminaries

For \( x \in \partial G \) define

\[
\delta_x = \min_{j \in I(x)} (x \cdot n_j - c_j),
\]

(7)

with the understanding that \( \delta_x = +\infty \) if \( I(x)^c = \{1, \ldots, N\} \setminus I(x) \) is empty. We see that \( \delta_x \) is always positive, possibly infinite. Consider any \( y \) with \( |y - x| < \delta_x \). Since the \( n_i \) are all unit vectors, it follows that \( y \cdot n_j > c_j \) for \( j \in I(x)^c \). Thus \( y \in G \) iff \( y \cdot n_i \geq c_i \) for \( i \in I(x) \). Moreover \( I(y) \subseteq I(x) \) for such \( y \). The following lemma provides the connection between the complementarity problem and the discrete projection, locally.

Lemma 1. Suppose \( w \) solves the complementarity problem (5) for \( x \in \partial G \) and \( v \in \mathbb{R}^n \). Then for all \( \epsilon > 0 \) with \( \epsilon|w| < \delta_x \) we have

\[ \Pi(x + \epsilon v) = x + \epsilon w. \]

Proof. The hypotheses imply \( |(x + \epsilon v) - x| < \delta_x \). Therefore for \( x + \epsilon v \in G \) it is sufficient that \( w \cdot n_i \geq 0 \) for \( i \in I(x) \), which is true from the features of the complementarity problem. If all \( \beta_i = 0 \) then \( w = v \) and \( x + \epsilon v = x + \epsilon w \in G \) so that \( \Pi(x + \epsilon v) = x + \epsilon w \), establishing our claim. Suppose instead that some \( \beta_j > 0 \). Then \( w \cdot n_j = 0 \), which implies \( x + \epsilon v \in \partial G \). We know that \( I(x + \epsilon v) = \{ i \in I(x) : w \cdot n_i = 0 \} \).

Let \( F = \{ i \in I(x) : \beta_i > 0 \} \). The complementarity conditions imply that \( F \subseteq I(x + \epsilon v) \) and therefore \( (x + \epsilon v) = (x + \epsilon w) = \sum_{i \in F} \epsilon \beta_i d_i \in d(x + \epsilon w) \). Hence \( \Pi(x + \epsilon v) = x + \epsilon w \), as claimed. \( \square \)

The following “uniform covering” result will be used for the proof of Theorem 3 below. Note that if \( G \) were compact, this would follow simply by considering the open cover of \( \partial G \) consisting of the balls \( \mathcal{N}_{\delta_x/K}(x) \).

Theorem 1. Suppose \( K > 0 \) is given. There exists a \( \delta > 0 \) with the property that for every \( y \notin G \) with \( d(y, G) < \delta \), there exists an \( x \in \partial G \) with \( |y - x| < \delta_x/K \).

In preparation for proving Theorem 1, for each \( I \subseteq \{1, \ldots, N\} \) we define the face \( F_I \) of \( \partial G \) for which the constraints \( i \in I \) are active:

\[ F_I = \{ x : x \cdot n_i = c_i \text{ for } i \in I, \text{ and } x \cdot n_j > c_j \text{ for } j \in I^c \}. \]

For \( \epsilon > 0 \) we define the \( \epsilon \)-core of \( F_I \) to be

\[ F_I^\epsilon = \{ x \in F_I : |y - x| < \epsilon \text{ and } y \cdot n_i = c_i \text{ for all } i \in I \text{ imply } y \cdot n_j > c_j \text{ for } j \in I^c \}. \]

In other words, \( F_I^\epsilon \) consists of those \( y \) in the affine hyperplane defined by the active constraints \( y \cdot n_i = c_i \), \( i \in I \), which are at least \( \epsilon \) away from any point where some other constraint \( (j \in I^c) \) is active.
We are only interested in those $F_I$ which are nonempty. Together these form a partition of $\partial G$:

$$\mathcal{F} = \{ F_I : F_I \neq \emptyset \}.$$  

We define a partial order on $\mathcal{F}$ by $F_I \sqsubset F_J$ if $I \subseteq J$ (and both $F_I$ and $F_J$ are nonempty). The maximal $F_I$ with respect to $\sqsubset$ are special. The following lemma explains why.

**Lemma 2.** Suppose $I \subseteq \{1, \ldots, N\}$ and $\epsilon > 0$.

a) If $F_I^* \epsilon F_I$ is nonempty then $0 < \inf_{x \in F_I^*} \delta_x$.

b) If a nonempty $F_I$ is maximal with respect to $\sqsubset$ then $F_I = F_I^*$.

c) If $x \in F_I \setminus F_I^*$ then there exists $z \in F_J$, some $F_I \sqsubset F_J$ with $|z - x| < \epsilon$.

**Proof.** For a) consider any $j \in I^c$. It suffices to show that $x \cdot n_j - c_j$ has a positive lower bound over $F_I^*$. Let $L_I$ be the the linear subspace of $\mathbb{R}^n$ defined by $v \cdot n_i = 0$ for all $i \in I$. Clearly any two points of $F_I$ differ by an element of $L_I$. There are two cases to consider.

First suppose that $L_I \perp n_j$. Then $x \cdot n_j - c_j$ is constant over $F_I^*$. Since this constant must be positive for any $x \in F_I^*$, it provides the desired lower bound.

Secondly, if $L_I \not\perp n_j$, there exists a unit vector $u \in L_I$ with $u \cdot n_j > 0$. Consider any $x \in F_I^*$ and $0 < \epsilon' < \epsilon$. Let $y = x - \epsilon'u$. Since $|y - x| < \epsilon$ and $x \cdot n_i = y \cdot n_i$ all $i \in I$, the definition of $F_I^*$ implies that $y \cdot n_j - c_j > 0$. Therefore

$$x \cdot n_j = (y + \epsilon' u) \cdot n_j > c_j + \epsilon' u \cdot n_j.$$  

Taking the supremum over $\epsilon' < \epsilon$ implies that $x \cdot n_j - c_j \geq \epsilon u \cdot n_j$ for all $x \in F_I^*$, establishing $\epsilon u \cdot n_j$ as the desired lower bound for a specific $j \in I^c$.

For b) note that for $F_I$ to be maximal means that there is no $z \in G$ such that $I \subseteq I(z)$. That being the case, consider any $y$ satisfying $y \cdot n_i = c_i$ for all $i \in I$. We claim that $y$ must also satisfy $y \cdot n_j > c_j$ for all $j \in I^c$. If, on the contrary, $y \cdot n_j \leq c_j$ for some $j \in I^c$, we could consider the line segment from $x$ to $y$. There would exist a $z$ on this line segment closest to $x$ at which $z \cdot n_k = c_k$ for some $k \in I^c$. At this $z$ we would have $z \cdot n_i = c_i$ for all $i \in I \cup \{k\}$ and $z \cdot n_j \geq c_j$ for all remaining $j$. But this violates the maximality of $F_I$, since $I \subseteq I \cup \{k\} \subseteq I(z)$. Therefore $y \cdot n_i = c_i$ for all $i \in I$ implies that $y \cdot n_j > c_j$ for all $j \in I^c$. In particular this means $F_I = F_I^*$.

For c), if $x \in F_I \setminus F_I^*$, then there exists $y$ with $|y - x| < \epsilon$, $y \cdot n_i = c_i$ for all $i \in I$ but $y \cdot n_j \leq c_j$ for some $j \in I^c$. Then, as above, along the line segment from $x$ to $y$ there exists a $z$ with $z \cdot n_i = c_i$ for all $i \in I \cup \{k\}$ and $z \cdot n_j \geq c_j$ for all remaining $j$. Then $z \in G$ and $I \cup \{k\} \subseteq I(z)$. Therefore $F_I \sqsubset F_J$ where $J = I(z)$. Since $z$ is on the line segment from $x$ to $y$, $|z - x| \leq |y - x| < \epsilon$.

We comment that more can be said about the partial ordering $\sqsubset$ of the proof. Specifically, $F_I$ is maximal if it is closed. The closure of $F_I$ is precisely the union of $F_I$ together with those $F_J$ for which $F_I \sqsubset F_J$. However, that is more than needed for our purposes. We can now prove Theorem 1.

**Proof.** Consider an $F_I$ which is maximal with respect to the partial ordering $\sqsubset$. By Lemma 2, $\delta_I = \frac{1}{K} \inf_{x \in F_I} \delta_x$ defines a positive value. For all $x \in F_I$, $\delta_I \leq \delta_x/K$. Thus if $d(y, F_I) < \delta_I$ then there exists $x \in F_I \subseteq \partial G$ for which $|y - x| < \delta_x/K$.

We now work down through the partial ordering of $\mathcal{F}$ to define $\delta_I$ recursively. Suppose that for all $F_I \sqsubset F_J$ we have defined $\delta_J > 0$ with the property that if $d(y, F_J) < \delta_J$ then there exists $x \in \partial G$ with $|y - x| < \delta_x/K$. Then we take

$$\epsilon_I = \frac{1}{2} \inf_{F_I \sqsubset F_J} \delta_J$$  

and define

$$\delta_I = \min \left( \epsilon_I, \frac{1}{K} \inf_{x \in F_I} \delta_x \right).$$

Lemma 2 tells us that $\delta_I > 0$. Suppose that $d(y, F_I) < \delta_I$. Then there is an $x \in F_I$ with $|y - x| < \delta_I$. If $x \in F_I^{*I}$ then $x \in \partial G$ and $|y - x| < \delta_I \leq \delta_x/K$.  

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Otherwise, if \( x \in F_I \setminus F^c_I \), Lemma 2 implies that there is \( z \in F_J \) for some \( F_I \cap F_J \) with \( |x - z| < \epsilon_I \leq \frac{1}{2} \delta_J \). Then \( |y - z| < \epsilon_I + \frac{1}{2} \delta_J \leq \delta_J \). By hypothesis, we know that this implies the existence of \( x \in \partial G \) with \( |y - x| < 2 \delta \).

Carrying out this construction from the maximal \( F_I \) downward through the ordering to the minimal, we eventually produce \( \delta_I > 0 \) for all nonempty \( F_I \). We now simply take \( \delta \) to be the smallest among the \( \delta_I \). Then if \( y \notin G \) has \( d(y, G) < \delta \), then \( d(y, F_I) < \delta_I \) for some nonempty \( F_I \), so that there does exist \( x \in \partial G \) with \( |y - x| < 2 \delta \), as claimed.

\[ \square \]

### 3 Necessity of the Complementarity Problems

We begin our arguments in this section by assuming the existence of a discrete projection \( y = \Pi(x) \) for each \( x \in \mathbb{R}^n \) and drawing out implications for the complementarity problems (5). By virtue of Assumption 2.1 we know that \( \Pi(\cdot) \) is continuous. The focus of our considerations is the velocity projection map (6). The existence of this limit was explained briefly in [4] for the particular case when \( G \) is a convex cone. We spell out an argument for the general case.

Consider \( x \in \partial G \) and a nonzero \( v \in \mathbb{R}^n \). Using the continuity of \( \Pi \), there exists \( \delta > 0 \) so that \( |\Pi(z) - x| \leq \delta_i \) whenever \( |z - x| \leq \delta \). Define \( \epsilon = \frac{\delta}{2 |v|} \), \( y = \Pi(x + \epsilon v) \) and \( w = \frac{1}{\epsilon}(y - x) \). Thus \( \Pi(x + \epsilon v) = x + \epsilon w \). Since \( |(x + \epsilon v) - x| = \frac{\delta}{\epsilon} < \frac{\delta}{2} < \frac{\delta}{2} \) it follows that

\[ |y - x| = |\epsilon w| = |\Pi(x + \epsilon v) - x| < \delta, \]

and therefore \( y \cdot n_i > c_i \) for any \( i \notin I(x) \). It follows that

\[ I(y) = \left\{ i \in I(x) : y \cdot n_i = c_i \right\} = \left\{ i \in I(x) : y \cdot n_i = x \cdot n_i \right\} = \left\{ i \in I(x) : w \cdot n_i = 0 \right\}. \tag{8} \]

We also know from the definition of \( \Pi \) that there exist \( \alpha_i \geq 0, i \in I(y) \),

\[ \epsilon(w - v) = \Pi(x + \epsilon v) - (x + \epsilon v) = \sum_{i \in I(y)} \alpha_i d_i. \tag{9} \]

Thus with \( \beta_i = \alpha_i/\epsilon \) for \( i \in I(y) \) and \( \beta_i = 0 \) for \( i \notin I(x) \setminus I(y) \) we have \( w = v + \sum_{i \in F} \beta_i d_i \) and

\[ \beta_i \geq 0; \quad w \cdot n_i \geq 0; \quad \text{and} \quad \beta_i(w \cdot n_i) = 0, \]

the last equality following from (8). In other words, \( w \) is a solution to the complementarity problem (5) for \( x \) and \( v \). We claim moreover that for all \( 0 < \Delta < \epsilon \), \( \Pi(x + \Delta v) = x + \Delta w \). To see this, observe that since \( |(x + \Delta w) - x| = |\Delta w| < \epsilon w = \delta_x \), and since \( w \cdot n_i = 0 \) for \( i \in I(x) \) we can say that \( x + \Delta w \in G \). Also, \( I(x + \Delta w) = \left\{ i \in I(x) : w \cdot n_i = 0 \right\} = I(y) \). Therefore, multiplying (9) by \( \Delta \),

\[ (x + \Delta w) - (x + \Delta v) = \Delta(w - v) = \sum_{i \in I(x + \Delta w)} \frac{\Delta}{\epsilon} \alpha_i d_i = d(x + \Delta w), \]

which shows that \( \Pi(x + \Delta v) = x + \Delta w \) as claimed. Thus for all \( 0 < \Delta < \epsilon \) we have

\[ \frac{\Pi(x + \Delta v) - x}{\Delta} = w. \]

This shows that the limit in (6) exists and \( w = \pi(x, v) \) is a solution of the complementarity problem (5). (If \( v = 0 \) it is trivial to check that \( 0 = \pi(x, 0) \) and solves the complementarity problem.) We have proven the following.

**Theorem 2.** Suppose that Assumption 2.1 holds and that for each \( x \in \mathbb{R}^n \) the discrete projection \( \Pi(x) \) exists. Then for any \( x \in \partial G \) and \( v \in \mathbb{R}^n \) the complementarity problem (5) has a solution \( w \) given by \( w = \pi(x, v) \), as defined by (6).
4 Sufficiency of the Complementarity Problems

We turn now to the converse, and assume that the complementarity problems (5) all admit solutions (any \( x \in \partial G, v \in \mathbb{R}^n \)). Our goal is to show the existence of the discrete projections \( \Pi(x) \). The first issues are uniqueness and continuity of solutions to the complementarity problems.

Lemma 3. Suppose Assumption 2.1 holds and that for every \( x \in \partial G \) and \( v \in \mathbb{R}^n \) the complementarity problem (5) has a solution \( w \). Then the solution \( w \) for a given \( x \) and \( v \) is unique and, for \( x \) fixed, the solution map \( v \mapsto w \) is Lipschitz continuous with the same constant \( K \) as in (4).

Proof. Consider \( x \in \partial G \) and two \( v_1 \) and \( v_2 \). Let \( w_1 \) and \( w_2 \) be (any) solutions of the respective complementarity problems. Then for a common \( \epsilon > 0 \) sufficiently small (less than both \( \delta_x/|v_1| \) and \( \delta_x/|v_2| \)) we would have from Lemma 1 that \( \Pi(x + \epsilon v_i) = x + \epsilon w_i \) for both \( i = 1, 2 \). The Lipschitz continuity (4) of \( \Pi \) implies

\[
\epsilon |w_1 - w_1| \leq K \epsilon |v_2 - v_1|.
\]

Considering \( v_2 = v_1 \) implies the uniqueness. The Lipschitz continuity of \( v \mapsto w \) is manifest after cancelling the \( \epsilon \). \( \square \)

As already noted, for \( v = 0 \) the complementarity problem is solved by \( w = 0 \). Therefore the Lipschitz continuity implies that the solution \( w \) for a given \( v \) satisfies

\[
|w| \leq K|v|,
\]

regardless of \( x \in \partial G \).

Theorem 3. Suppose Assumption 2.1 holds and that for every \( x \in \partial G \) and \( v \in \mathbb{R}^n \) there exists a solution \( w \) of the complementarity problem (5). Then the discrete projection \( \Pi(x) \) exists for all \( x \in \mathbb{R}^n \).

Proof. By virtue of Theorem 4.4 of [10] it is sufficient to prove that, for some \( \delta > 0 \), \( \Pi(y) \) exists for all \( y \) with \( d(y, G) < \delta \). Consider the \( \delta > 0 \) of Theorem 1, using the Lipschitz constant \( K \), and suppose \( y \in G \) with \( d(y, G) < \delta \). There exists \( x \in \partial G \) with \( |y - x| < \delta_x/\epsilon \) and \( \epsilon = K \). Take \( v = y - x \) and let \( w \) be the solution of the complementarity problem for \( x, v \). It follows that \( |w| < K|v| < \delta_x \), and therefore by Lemma 1, \( \Pi(y) = \Pi(x + v) \) exists and is given by \( x + w \). \( \square \)

5 A Coercivity Condition for Existence

Based on the above, any condition which is sufficient for the solvability of the complementarity problems (5) is sufficient for the existence of the projection \( \Pi \) and hence for the existence of solutions to the Skorokhod problem. A standard sufficient condition for the existence of a solution \( x \in \mathbb{R}^d \) to a complementarity problem of the form

\[
\begin{align*}
x & \geq 0; & \text{(componentwise)} \\
y & \geq 0 & \text{where} \ y = Mx + q; \text{ and} \\
y \cdot x & = 0,
\end{align*}
\]

is that the coefficient matrix \( M \) be coercive (i.e. \( u \cdot Mu > 0 \) for all \( u \neq 0 \)). See for instance Chapter 1 of [11], Corollary 4.3 and Theorem 5.5. (This coercivity condition also implies uniqueness of the solution \( x \) and \( y \). In our context we already know the uniqueness of \( y \) as a consequence of Assumption 2.1.) Our problems (5) are of this form using \( M = [m_i \cdot d_j]_{i,j \in I(x)} \), which we will denote \( M = N^T D_I, I = I(x) \) using the notation of [8]. Thus for our problems (5) the coercivity of \( N^T D_I \) for each \( I = I(x), x \in \partial G \), is a sufficient condition for the existence of \( \Pi \).

Theorem 4. Suppose that Assumption 2.1 holds and that \( N^T D_I \) is coercive for each \( I = I(x), x \in \partial G \). Then the discrete projection \( \Pi(x) \) exists for all \( x \).
Note that coercivity of $N_I^T D_I$ is equivalent to positive definiteness of
\[
\frac{1}{2}(N_I^T D_I + D_I^T N_I),
\]
making it rather easy to check. A consequence of the coercivity condition is that \(\{n_i : i \in I\}\) is linearly independent. Thus in \(\mathbb{R}^n\) we can only hope to apply Theorem 4 if no more than \(n\) of the constraints (1) are active at each \(x \in G\).

On the other hand, coercivity allows a direct proof of Lipschitz continuity of the solution map \(q \mapsto x\) of the complementarity problem (10). Suppose \((q_i, x_i, y_i)\) are solution triples for \(i = 1, 2\) and write \(\Delta x = x_2 - x_1\) and likewise for \(\Delta y\) and \(\Delta q\). It follows that
\[
\Delta y = M \Delta x + \Delta q
\]
\[
\Delta x \cdot \Delta y = \Delta x \cdot M \Delta x + \Delta x \cdot \Delta q.
\]
Since \(x_i, y_i = 0\) and all coordinates of \(x_i\) and \(y_i\) are nonnegative, it follows that \(\Delta x \cdot \Delta y = -(x_2 \cdot y_1 + x_1 \cdot y_2) \leq 0\). Therefore
\[
\Delta x \cdot M \Delta x \leq -\Delta x \cdot \Delta q.
\]
Since \(M\) is coercive there is a positive constant \(c > 0\) such that \(u \cdot Mu \geq c|u|^2\) for all \(u\). Thus
\[
c|\Delta x|^2 \leq |\Delta x||\Delta q|,
\]
which implies that
\[
|\Delta x| \leq c^{-1}|\Delta q|.
\]
As a consequence, \(|\Delta y| \leq (1 + \|M\||c^{-1}|)|\Delta q|\). Using this and the existence of solutions to (10), we see that the conclusions of Lemma 3 remain true if we replace its hypotheses by the coercivity of \(M = N_I^T D_I\) for all \(I = I(x), x \in \partial G\).

References


