I. Rolle’s Theorem

A. Rolle’s Theorem

Let \( f \) be a function that satisfies the following three hypotheses:

1. \( f \) is continuous on the closed interval \([a,b]\).
2. \( f \) is differentiable on the open interval \((a,b)\).
3. \( f(a) = f(b) \)

Then there is a number \( c \) in \((a,b)\) such that \( f'(c) = 0 \).

B. Examples

1. Verify that \( f(x) = x\sqrt{x+6} \) satisfies the three hypotheses of Rolle’s Theorem on the interval \([-6,0]\). Then find all numbers \( c \) that satisfy the conclusion of Rolle’s Theorem.

2. The function \( f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases} \) is zero at \( x=0 \) and \( x=1 \) and differentiable on \((0,1)\), but its derivative on \((0,1)\) is never zero. How can this be? Doesn’t Rolle’s Theorem say the derivative has to be zero somewhere in \((0,1)\)? Explain why this doesn’t violate Rolle’s Theorem.
II. The Mean Value Theorem

A. The Mean Value Theorem

Let $f$ be a function that satisfies the following hypotheses:
1. $f$ is continuous on the closed interval $[a,b]$.
2. $f$ is differentiable on the open interval $(a,b)$.

Then there is a number $c$ in $(a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$
or, equivalently $f(b) - f(a) = f'(c)(b - a)$.

B. Physical Interpretation

1. The Mean Value Theorem says that at some interior point the instantaneous rate of change must equal the average rate of change.
2. There is an interior point where the slope of the tangent line must be equal to the slope of the secant line, i.e., the tangent line is parallel to the secant line.

C. Examples

1. The graph of $f(x) = x + \frac{4}{x}$ is below. Graph the secant line through the points $(1,5)$ and $(8,8.5)$ on the graph below.

Find the number $c$ that satisfies the conclusion of the Mean Value Theorem for this function $f$ and the interval $[1,8]$. Then graph the tangent line at the pt $(c,f(c))$ and notice that it is parallel to the secant line.
2. Verify that $f(x) = e^{-2x}$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0,3]$. Then find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.

III. Additional Theorems and Corollary

A. Theorem: If $f'(x) = 0$ for all $x$ in an interval $(a,b)$, then $f$ is a constant.

B. Corollary: If $f'(x) = g'(x)$ for all $x$ in an interval $(a,b)$, then $f - g$ is constant on $(a,b)$; i.e., $f(x) = g(x) + c$ where $c$ is a constant.
IV. Additional Examples

1. Show that the equation $3x - 2 + \cos\left(\frac{x}{2}\right) = 0$ has exactly one real root.

2. For what values of $a$, $m$, and $b$ does the function $f(x) = \begin{cases} 
3 & ; x = 0 \\
-x^2 + 3x + a & ; 0 < x < 1 \\
mx + b & ; 1 \leq x \leq 2
\end{cases}$ satisfy the hypotheses of the Mean Value Theorem on the interval $[0,2]$?