Math 1205 Calculus/Sec. 3.3 Rates of Change in the Natural and Social Sciences

I. Recall:

A. Average Rate of Change

1. The average rate of change of \( y = f(x) \) wrt \( x \) over the interval \([x_1, x_2]\) is
   \[
   \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \text{where } h = x_2 - x_1.
   \]

2. Difference Quotient:
   \[
   \frac{f(x_2 + h) - f(x_1)}{h}, \text{where } h = x_2 - x_1
   \]

3. Geometrically, the average rate of change is the slope of the secant line connecting the pts \((x_1, f(x_1))\) and \((x_2, f(x_2))\).

4. Physically, the average rate of change is the average velocity.

B. (Instantaneous) Rate of Change

1. The rate of change of \( y = f(x) \) wrt \( x \) at \( x = x_1 \) is
   \[
   \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} , \text{where } h = x_2 - x_1
   \]
   provided these limits exist.

2. Derivative at a point \( x = x_1 \): \( f'(x_1) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} \), provided the limit exists.

3. Geometrically, the rate of change is the slope of the tangent line at \( x = x_1 \).

4. Physically, the rate of change is the velocity at \( x = x_1 \).

II. Physics

A. Rectilinear motion / Free Fall (Air resistance will be ignored)

1. Position Function: \( s(t) \) – position as a function of time of a particle that is moving in a straight line.

2. Velocity function:
   a. Velocity is the rate of change of displacement \( s \), with respect to time
   
   b. Formula: \( v(t) = \frac{ds}{dt} = s'(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} \)
c. Since the derivative graphically represents the slope of the tangent line at \( t_0 \),

\[ m_{\text{tan}} > 0 \Rightarrow s'(t_0) > 0 \Rightarrow v(t_0) > 0 \Rightarrow \text{graph of } s(t) \text{ is rising} \Rightarrow \text{position is increasing} \Rightarrow \text{object is moving to the right / up}. \]

\[ m_{\text{tan}} < 0 \Rightarrow s'(t_0) < 0 \Rightarrow v(t_0) < 0 \Rightarrow \text{graph of } s(t) \text{ is falling} \Rightarrow \text{position is decreasing} \Rightarrow \text{object is moving to the left / down}. \]

3. Acceleration function: \( a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = s''(t) = \lim_{D \to 0} \left[ \frac{v}{D} \right] \)

a. When \( a(t) > 0 \Rightarrow \text{velocity is increasing} \)

1. \( v(t) > 0 \Rightarrow \text{object speeds up} \)
2. \( v(t) < 0 \Rightarrow \text{object slows down} \)

b. When \( a(t) < 0 \Rightarrow \text{velocity is decreasing} \)

1. \( v(t) > 0 \Rightarrow \text{object slows down} \)
2. \( v(t) < 0 \Rightarrow \text{object speeds up} \)

c. Summary of the relationship between the signs of velocity and acceleration

1.) When velocity and acceleration have the same algebraic sign, the object speeds up.
2.) When velocity and acceleration have the different algebraic signs, the object slows down.

4. Displacement: \( s = s(t + \Delta t) - s(t) \) (change in position – Not total distance)

5. Speed: \( |v(t)| = \left| \frac{ds}{dt} \right| \); Speed \( \geq 0 \)!!

B. Linear density, \( \square \)

1. If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length (\( \square = \text{m/}l \)) and is measured in kilograms per meter.

2. If the rod is not homogeneous, its mass can be measured from its left end to a pt \( x \) on the rod as \( m = f(x) \). The mass of the rod that lies between \( x = x_1 \) to \( x = x_2 \) is given by \( \square m = f(x_2) - f(x_1) \), so the average density of that part of the rod is

\[
\text{average density} = \frac{\square m}{\square x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

3. The linear density \( \square \) at \( x_1 \) is the rate of change of mass wrt length, i.e. the linear density is the derivative of mass wrt length.

\[ \square = \frac{dm}{dx} \]
III. Examples
1. Suppose \( s(t) = t^2 - 4t - 3 \) gives the position of a body moving on a coordinate line for \( 0 \leq t \leq 6 \), where \( t \) is measured in seconds and \( s \) in meters.
   a. Determine the position of the particle at the endpoints.

   b. Determine the displacement.

   c. Determine the average velocity of the particle over the interval \([0,6] \)

   d. Determine velocity after 1s and after 4s.

   e. Determine when the particle changes direction / when the particle is at rest.

   f. Determine when the particle is moving forward (i.e., in a positive direction).

   g. Determine the total distance traveled by the particle on the interval \([0,6] \).

   h. Draw a diagram to represent the motion of the particle.
2. A projectile is fired straight up from the ground. Its distance from the ground after $t$ seconds is: $s(t)=-16t^2+200t$ (measured in feet)
   a. Determine the velocity at $t=4$ sec and at $t=9$ sec.
   b. Determine when the projectile reaches its maximum height.
   c. Determine the maximum height the projectile obtains.
   d. Determine when the projectile hits the ground.
   e. Determine the impact velocity of the projectile.

3. The mass of a part of a rod is given by $m = \sqrt{x}$, where $x$ is measured in meters and $m$ in kilograms.
   a. Find the average density of the part of the rod given by $1 < x \leq 2$.
   b. Find the linear density at $x=1$. 
4. The quantity of charge $Q$ in coulombs (C) that has passed through a pt in a wire up to time $t$ (measured in seconds) is given by $Q(t)=t^3-2t^2+6t+2$. Find the current $I$, when $t=0.5$ s if the current $I$ is the rate at which charge flows through a surface, i.e., $I = \frac{dQ}{dt}$.

5. A rock is tossed straight up from a cliff 112 ft above the ground. Its distance from the ground below after $t$ seconds is: $s(t)=-16t^2+96t+112$ (measured in ft)
   a. Determine the velocity and the acceleration functions.
   b. Determine the initial velocity of the rock
   c. Determine the maximum height the rock obtains.
   d. Determine the rock's impact velocity.

6. A spherical balloon is being inflated. Find the rate of increase of the surface area ($S=4\pi r^2$) wrt the radius $r$ when $r$ is 2 ft.