Math 1205 Calculus – Sec. 2.5: Continuity

I. Definitions
   A. **Def**: A function $f$ is continuous at an interior point $x = c$ of its domain if $\lim_{x \to c} f(x) = f(c)$.

   B. **Def**: A function $f$ is continuous at a left endpoint $x = a$ of its domain if $\lim_{x \to a^-} f(x) = f(a)$.

   C. **Def**: A function $f$ is continuous at a right endpoint $x = b$ of its domain if $\lim_{x \to b^+} f(x) = f(b)$.

II. Continuity Test
    A function $f(x)$ is continuous at $x = c$, if and only if:
    1. $f(c)$ exists
    2. $\lim_{x \to c} f(x)$ exists
    3. $\lim_{x \to c} f(x) = f(c)$

*Note*: When dealing with endpoints we can replace #2 & #3 with left hand or right hand limits.

II. Types of Discontinuities:
   1. Removable discontinuities:
      
      Fails step 3 of the continuity test
      
      $\lim_{x \to c} f(x) \neq f(c)$
      
      $f(c)$ may or may not exist.

      Fails step 1 of the continuity test but the limit exists.
II. Jump Discontinuity

Jump discontinuity at \( x=0 \)

2. Jump Discontinuity

3. Infinite Discontinuities

Infinite discontinuity at \( x=0 \)

Fails step 2 of the continuity test because
\[
\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x).
\]

Fails step 2 of the continuity test because
\[
\lim_{x \to c^-} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = \pm \infty.
\]

f(c) may or may not exist.

III. Continuity on an Interval

A. Def: A function is continuous on an interval if it is continuous at every number in the interval. (If \( f \) is defined on only one side of an endpt of the interval, we understand continuous at the endpt to mean continuous from the right or continuous from the left.)

B. Examples

1. Continuous on a closed interval \([a,b]\) / Continuous Function at the Endpoints of an interval:
   \[
   f(x) = \sqrt{9 - x^2}.
   \]
   Continuous on its domain/ at its endpts

   \[
   \begin{align*}
   x & \quad \text{at its endpts} \\
   -3 & \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3
   \end{align*}
   \]
2. Continuous on an open interval $(a,b)$

3. Continuous on an semi-open interval $[a,b)$

D. Continuity on its domain

\[ y = \sqrt{x} \text{ is continuous on its entire domain} \]

IV. Theorems
A. If $f$ and $g$ are both continuous at $x = a$ and $c$ is a constant, then the following are continuous at $x = a$:

1. $f + g$ and $f - g$
2. $cf$
3. $fg$

4. \( \frac{f}{g} \) (provided $g(a) \neq 0$)
B. Any polynomial is continuous everywhere; i.e., continuous on \( \mathbb{R} = (\emptyset , \infty ) \)

C. Any rational function is continuous wherever it is defined; i.e., continuous on its domain. (its denominator is not equal to zero).

D. Every root function, trigonometric function, inverse trigonometric function, exponential function, and logarithmic function is continuous on its domain.

E. If \( f \) is continuous at \( c \) and \( \lim_{x \to a} g(x) = c \), then \( \lim_{x \to a} f(g(x)) = f(c) \). In other words,
\[
\lim_{x \to a} f(g(x)) = f\left( \lim_{x \to a} g(x) \right).
\]

F. If \( g \) is continuous at \( a \), and \( f \) is continuous at \( g(a) \), then the composite fn \( f \circ g \) given by
\[
(f \circ g)(x) = f(g(x))
\]
is continuous at \( x = a \).

V. Examples

A. Given the function below, discuss its continuity on the interval \([0, 4]\).

\[
\begin{array}{c|c}
\hline
x & y \\
\hline
0 & 0.5 \\
1 & 1.0 \\
2 & 2.5 \\
3 & 3.0 \\
4 & 3.5 \\
\hline
\end{array}
\]

Note: The fn is continuous except at the values found above.
B. Given \( f(x) = \begin{cases} 
  x^3 & ; \ x \leq -1 \\
  x & ; -1 < x < 2 \\
  \frac{x}{(x-3)^2} & ; \ 2 \leq x \leq 4
\end{cases} \)

Discuss the continuity at the breaks in the domain, i.e., at \( x=1,2,4 \). Are there any pts of discontinuity that we miss by just looking at the breaks in the domain?

C. State the intervals on which the function is continuous.

1. \( f(x) = \sqrt{x^2 - 4} \)  
2. \( h(x) = \frac{x \cos(x^2)}{1 + x^4} \)

3. \( g(x) = \frac{x^2 + x}{x^3 - x} \)

4. \( y = \sec(x) \)
5. Find the value for c that will make \( f(x) = \begin{cases} \frac{cx}{2} + 2 & ; x < 2 \\ \frac{c}{x^2} & ; x \geq 2 \end{cases} \) continuous on \(( , )\).

VI. Continuous Extension to a Point:

A. Process

Here we are basically explaining how we can deal with a removable point of discontinuity.

Given a function \( f(x) \) that has a removable point of discontinuity at \( x = c \).

Then define \( F(x) = \begin{cases} f(x) & ; \text{if } x \text{ is in the domain of } f \\ \lim_{x \to c} f(x) = L & ; \text{if } x = c \end{cases} \)

We now have a continuous function at all points.

B. Examples

Find the discontinuities of the functions below. If the function has a removable discontinuity, define a new function \( F(x) \) that agrees with \( f(x) \) everywhere except the discontinuity and is continuous at the discontinuity.

1. \( f(x) = \frac{4x - x^2}{2 \sqrt{x}} \)

2. \( f(x) = \frac{x^2 + x \sqrt{12}}{x \sqrt{3}} \)
VII. Intermediate Value Theorem:

A. Theorem

Suppose that \( f \) is continuous on the closed interval \([a,b]\) and let \( N \) be any number between \( f(a) \) and \( f(b) \). Then \( \exists \) a number \( c \) in \((a,b)\) s.t. \( f(c)=N \).

B. Example

Show that \( f(x)=\cos(x)-x \) has a root in the interval \([0, 1]\).