Math 1205 Calculus – Sec. 2.4: The Precise Definition of a Limit

I. Review
   A. Informal Definition of Limit
      1. **Def**: Let $f(x)$ be defined on an open interval about $a$ except possibly at $a$ itself. If $f(x)$ gets arbitrarily close to $L$ (as close to $L$ as we like) for all $x$ sufficiently close to $a$, we say that $f$ approaches the limit $L$ as $x$ approaches $a$ and we write: \[ \lim_{x \to a} f(x) = L \] ("the limit of $f(x)$, as $x$ approaches $a$, equals $L".")

      2. **Note**: (1) $x \to a$ means that you approach $x=a$ from both sides of $a$.
         (2) $f(a)$ does not have to be defined.

   B. Solving an inequality
      1. **Fill in the blanks**: To say that $|x-3| < 1$ means that $x$ is less than _______ units from _______.
      2. Solve $|x-3| < 1$

II. Introduction
   A. Consider the function $f(x) = -2x+5$. How close to $a=1$, must we hold $x$ to be sure that $f(x)$ lies within 1.5 units of $f(a)=3$?

   ![Graph of $f(x) = -2x+5$]

   How do we determine the value of delta, $\delta$?
   Method 1: Set $f(x) = L+\varepsilon$ and set $f(x) = L-\varepsilon$ and solve for $x_1 = a-\delta$ and $x_2 = a+\delta$ and then determine $\delta$
Method 2: What we want to know is: When is $|f(x) - L| < 1.5$?

$$|f(x) - 3| < 1.5$$
$$\Rightarrow |(-2x+5)-3| < 1.5$$
$$\Rightarrow |-2x+2| < 1.5$$
$$\Rightarrow -1.5 < -2x + 2 < 1.5$$
$$\Rightarrow -3.5 < -2x < -0.5$$
$$\Rightarrow 1.75 > x > 0.25 \Rightarrow 0.25 < x < 1.75 \Rightarrow x \in (0.25, 1.75)$$

$$\Rightarrow \delta = |0.25 - 1| = |1.75 - 1| = 0.75$$

III. The Precise Definition of a Limit

A. Def\textsuperscript{n}: Let $f$ be a fn defined on some open interval that contains the number $a$, except possibly at $a$ itself. Then we say that the limit of $f(x)$ as $x$ approaches $a$ is $L$, and we write \( \lim_{x \to a} f(x) = L \) if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$

Another way of writing the last line of this definition is: if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

Note: We have replaced the imprecise descriptions (i.e. sufficiently close and arbitrarily close) in the informal definition of limits with the values epsilon and delta.

B. If the value of $\varepsilon$ is specified

1. Graphical Approach

   a. Use the graph of $f(x) = x + 1$ to find a number $\delta$ such that if $0 < |x - 8| < \delta$ then $|\sqrt{x+1} - 3| < 1$, i.e., Use the graph of $f(x) = \sqrt{x+1}$, $L = 3$, and $\varepsilon = 1$ to determine $\delta$.

Now we need to find a delta neighborhood about 2 that will fit inside the open interval $(3,15)$.

The largest delta that will work is the smaller of the 2 distances from $a = 8$. \( \therefore \) we choose $\delta < \underline{__________}$
2. Algebraic Approach  
   a. Steps:
      1. Solve the inequality $|f(x)-L|<\varepsilon$ to find an open interval $(a,b)$ about $x_0$ on which the inequality holds for all $x \neq x_0$ (i.e. $\forall x \neq x_0$) 
      2. Find a value of $\delta>0$ that places the open interval $(x_0-\delta, x_0+\delta)$ centered at $x_0$ inside the interval $(a,b)$. The inequality $|f(x)-L|<\varepsilon$ will hold $\forall x \neq x_0$ in this delta-interval about $x_0$. 
   b. Example
      1. For the limit $\lim_{x \to 8} \sqrt{x+1} = 3$ illustrate the definition by finding the values of $\delta$ that correspond to $\varepsilon=1$. 

2. The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6cm. How closely must we measure the height, $h$, in order to measure out 1 L (1000 cm$^3$) with an error of no more than 1% (i.e. 10 cm$^3$)? (Use: $V=\pi r^2 h$)
C. If the value of $\varepsilon$ is not specified

1. An example of a proof worked out

Prove that $\lim_{x \to 1} (3x + 5) = 8$

a. Preliminary analysis of the problem (guessing a value for $\delta$)

Given $\varepsilon > 0$, find $\delta > 0$ s.t. $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$$0 < |x - 1| < \delta \Rightarrow |(3x + 5) - 8| < \varepsilon$$

$$\Rightarrow |3x - 3| < \varepsilon$$

$$\Rightarrow 3|x - 1| < \varepsilon$$

$$\Rightarrow |x - 1| < \frac{\varepsilon}{3}$$

This suggests that we choose $\delta = \frac{\varepsilon}{3}$

b. Proof (showing that this $\delta$ works)

Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{3}$. If $0 < |x - 1| < \delta$, then

$$|3x + 5) - 8| = |3x - 3| = 3|x - 1| < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Thus $|3x + 5) - 8| < \varepsilon$ whenever $0 < |x - 1| < \delta$

$\therefore$, by the definition of a limit, $\lim_{x \to 1} (3x + 5) = 8$

2. Example

Prove that $\lim_{x \to 2} (4x - 1) = 7$
D. More Definitions

1. **Def\textsuperscript{a} of Left-Hand Limit:** \( \lim_{x \to a^-} f(x) = L \) if for every number \( \varepsilon > 0 \) there is a corresponding number \( \delta > 0 \) such that \( |f(x) - L| < \varepsilon \) whenever \( a - \delta < x < a \)

2. **Def\textsuperscript{a} of Right-Hand Limit:** \( \lim_{x \to a^+} f(x) = L \) if for every number \( \varepsilon > 0 \) there is a corresponding number \( \delta > 0 \) such that \( |f(x) - L| < \varepsilon \) whenever \( a < x < a + \delta \)

3. **Infinite Limits:**
   a. Let \( f \) be a fn defined on some open interval that contains the number \( a \), except possibly at \( a \) itself. Then \( \lim_{x \to a} f(x) \to \infty \) means that for every positive number \( M \) there is a corresponding positive number \( \delta \) such that \( f(x) > M \) whenever \( 0 < |x - a| < \delta \)
   b. Let \( f \) be a fn defined on some open interval that contains the number \( a \), except possibly at \( a \) itself. Then \( \lim_{x \to a} f(x) \to -\infty \) means that for every negative number \( N \) there is a corresponding positive number \( \delta \) such that \( f(x) < N \) whenever \( 0 < |x - a| < \delta \)

IV. Extra Examples

A. Use the graph of \( f(x) = x^2 \) to find a number \( \delta \) such that \( |x^2 - 4| < 0.5 \) whenever \( |x - 2| < \delta \)

Now we need to find a delta neighborhood about 2 that will fit inside the open interval \((3.5, 4.5)\).

The largest delta that will work is the smaller of the 2 distances from \( x_0 = 2 \). \( ∴ \) we choose \( \delta < \) _____

\[
\begin{align*}
d_1 &= \underline{\quad} \\
d_2 &= \underline{\quad} \\
\{ &\delta < \underline{\quad} \\
\end{align*}
\]
B. Now resolve the above problem using an inequality. For the limit \( \lim_{x \to 2} x^2 = 4 \) illustrate the definition by finding the values of \( \delta \) that correspond to \( \varepsilon = 0.5 \).

C. Use the graph of \( f(x) = 4x + 2 \), \( L = -6 \) and \( \varepsilon = 1 \) to determine \( \delta \).

D. Now resolve the above problem using an inequality. For the limit \( \lim_{x \to -2} (4x + 2) = -6 \) illustrate the definition by finding the values of \( \delta \) that correspond to \( \varepsilon = 1 \).