Abbreviations: wrt = with respect to  \( \forall \) = for all  \( \exists \) = there exists
\( \therefore \) = therefore  Def = definition  Th = Theorem
sol = solution  \( \perp \) = perpendicular  iff or \( \iff \) = if and only if
pt = point  fn = function  eq = equation
\( \in \) = is an element of  st = such that

I.  Introduction:
  A.  Recall the following formulas from algebra:
    1.  Slope of a line: \( m = \frac{y_2 - y_1}{x_2 - x_1} \)
    2.  Average Speed: \( s = \frac{\text{amount of distance covered}}{\text{length of interval}} \)

B.  Examples
   1.  What is the slope of the line between the points (2,4) and (4,16)?

      Interpret the meaning of this slope.

   2.  A rock falls from the top of a 150 ft cliff. What is its average speed
       a.  during the first 2 seconds of fall?
       b.  during the 1 sec. interval between second 1 and second 2?
       Use \( s(t) = -16t^2 + 150 \)

II.  Secant Line
A.  Def:  The secant line to the curve \( y = f(x) \) is the line connecting two points,
\( (a, f(a)) \) and \( (b, f(b)) \), on the curve.

B.  Slope of a secant line: \( m_{sec} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a} \)
C. Alternative form of the slope of a secant line: 

\[ m_{\text{sec}} = \frac{f(a + h) - f(a)}{h} \]

where \( h = b - a \). When we write it this way, it is called the difference quotient of \( f \) at \( a \).

III. Average Rate of Change

A. Definition: The average rate of change of \( y = f(x) \) wrt \( x \) over the interval \([x_1, x_2]\) is

\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \text{where } h = x_2 - x_1
\]

B. Note: The slope of the secant line is the geometrical representation of the average rate of change.

IV. Tangent Line

A. What is a Tangent Line?
   a. Wrt a Circle: (1) A line is tangent to a circle at a pt \( P \) if the line \( L \) passes through \( P \) ⊥ to the radius at pt \( P \). (2) Euclid states that a tangent is a line that intersects the circle once and only once.
   b. Wrt a Line: The tangent is the line itself
   c. Wrt a Curve: (1) “Draw the best circle inside of the curve” (circle of curvature), (2) a tangent line can touch the curve in more than one place, (3) Definition: The tangent line to the curve \( y = f(x) \) at the pt \( P \) is the line \( L \) though \( P \) whose slope is the limit as pt \( Q \) approaches pt \( P \) of the slope of the secant line (from either side)
B. Calculating the Tangent Line

1. Using the graph: Draw the tangent line and then pick another point on the tangent and calculate this tangent using the point slope equation for a line. The main drawbacks for this method are the accuracy of your drawing and the grid lines on the graph paper.

2. Using the Function: Pick a second point Q on the graph of the fn y=f(x) close to pt P. Connect the points to form the secant line. As the change in x gets very small (close to zero) the secant line will become the tangent line. The main drawback to this method is if the curve takes a drastic turn. **Make a chart to determine the slope of the secant lines as pt Q gets closer to pt P.

Pt Q approaching P from the right.  (Secant line in red, tangent line in blue)
C. Examples

1. Determine the tangent line to the curve of \( y = x^2 + 1 \) at the pt \((1,2)\).

<table>
<thead>
<tr>
<th>Pt Q approaching P from the right.</th>
<th>Slope of secant line</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1,2) &amp; (2,5))</td>
<td></td>
</tr>
<tr>
<td>((1,2) &amp; (1.5,3.25))</td>
<td></td>
</tr>
<tr>
<td>((1,2) &amp; (1.1,2.21))</td>
<td></td>
</tr>
<tr>
<td>((1,2) &amp; (1.01,2.0201))</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pt Q approaching P from the left.</th>
<th>Slope of secant line</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1,2) &amp; (0,1))</td>
<td></td>
</tr>
<tr>
<td>((1,2) &amp; (0.5,1.25))</td>
<td></td>
</tr>
<tr>
<td>((1,2) &amp; (0.9,1.81))</td>
<td></td>
</tr>
<tr>
<td>((1,2) &amp; (0.99,1.9801))</td>
<td></td>
</tr>
</tbody>
</table>

**Now average the slopes of the two closest points to find the slope of the tangent line.

\[
m_{\text{tan}} = \underline{\text{___________}}
\]

The tangent line to the curve of \( y = x^2 + 1 \) at the pt \((1,2)\) is \underline{\text{_______________}}
2. The profits of a small company for each of the first five years of its operation are given in the following table:

<table>
<thead>
<tr>
<th>Year</th>
<th>Profit in the $1000s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1991</td>
<td>6</td>
</tr>
<tr>
<td>1992</td>
<td>27</td>
</tr>
<tr>
<td>1993</td>
<td>62</td>
</tr>
<tr>
<td>1994</td>
<td>111</td>
</tr>
<tr>
<td>1995</td>
<td>174</td>
</tr>
</tbody>
</table>

a. Find the slopes of the secant lines wrt 1993.

b. Estimate the slope of the tangent line for 1993.

c. Interpret this slope.

V. Velocity

A. Average Velocity

Another application of the average rate of change is the average velocity. Average velocity is calculated over a specific time interval \([t_1, t_2]\)

\[
\text{Ave Velocity} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}
\]

B. Instantaneous Velocity

(Instantaneous) Velocity is the velocity of an object at a specific time value, not the velocity over a time interval.

C. Example

1. The distance to the right of the starting position of an object after \(t\) sec. is \(3t^2\) ft.

   a. What is the average velocity for each of the following time intervals?

<table>
<thead>
<tr>
<th>Time interval</th>
<th>Average Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>[4,5]</td>
<td></td>
</tr>
<tr>
<td>[4.9,5]</td>
<td></td>
</tr>
<tr>
<td>[4.99,5]</td>
<td></td>
</tr>
<tr>
<td>[5,6]</td>
<td></td>
</tr>
<tr>
<td>[5,5.1]</td>
<td></td>
</tr>
<tr>
<td>[5,5.01]</td>
<td></td>
</tr>
</tbody>
</table>

b. How fast is the object moving at 5 seconds? (Hence, what is its velocity at 5 sec?)
VI. Summary
A. Average Rate of Change
- The average rate of change of \( y = f(x) \) wrt \( x \) over the interval \([x_1, x_2]\) is
  \[
  \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h},\text{where } h = x_2 - x_1.
  \]
- Geometrically, the slope of the secant line represents the average rate of change.
- Physically, the average velocity represents the average rate of change.

B. (Instantaneous) Rate of Change
- The rate of change of \( y = f(x) \) wrt \( x \) at \( x = x_1 \) is
  \[
  \lim_{{\Delta x \to 0}} \left( \frac{\Delta y}{\Delta x} \right) = \lim_{{x_2 \to x_1}} \left( \frac{y_2 - y_1}{x_2 - x_1} \right) = \lim_{{x_2 \to x_1}} \left( \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right) = \lim_{{h \to 0}} \left( \frac{f(x_1 + h) - f(x_1)}{h} \right),\text{where } h = x_2 - x_1
  \]
- Geometrically, the slope of the tangent line represents the rate of change.
- Physically, the velocity represents the rate of change.