Math 1205 Calculus – Taylor Polynomials

I. Introduction
   A. Review
      1. In section 3.10, it was shown that if we zoomed in on a differentiable function, the
tangent line lies on top of the curve near the point of tangency. The tangent line at
\((a, f(a))\) can be used as an approximation to the curve \(y = f(x)\) when \(x\) is near \(a\). The
equation of the tangent line at \((a, f(a))\) is 
\[ y = f(a) + f'(a)(x - a) \]
\[ \Rightarrow \]
\[ y = f(a) + f'(a)(x - a). \]
The approximation \(f(x) = f(a) + f'(a)(x - a)\) is called the
linear approximation of \(f\) at \(a\). The linearization \(L(x)\) is also called the \(1^\text{st} \) order Taylor
Polynomial Approximation for \(f(x)\) near \(x = a\).

2. Factorial notation, \(n!\)
   a. \(n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1\)
   b. For example: \(5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120\)
   c. \(0! = 1\)

B. Development of the formula
   1. Let \(f\) be a function defined on some domain containing the point \(x = a\). Sometimes we
   can represent a more complicated function in terms of a polynomial function. This can
   be very helpful as polynomial functions can be easily examined. Suppose we want to
   use a polynomial of a certain fixed degree to approximate the function \(f\) as well as we
   can in some neighborhood of the point of interest \(x = a\). The polynomial that
   approximates our curve at \(x = a\) will look like:
\[ P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n, \]
where
   \(c_0, c_1, c_2, \ldots, c_n\) are arbitrary constants.

2. Finding the values of \(c_0, c_1, c_2, \ldots, c_n\):
   a. The polynomial \(P_n\) and the function \(f\) need to have the same value at \(x = a\)
   \[ \Rightarrow c_0 = f(a). \]
   b. The polynomial and function need to have the same slope at \(x = a\). (i.e., set the
derivatives of \(f\) and \(P_n\) equal to each other) \[ \Rightarrow c_1 = f'(a). \]
   c. The polynomial and function need to have the same second derivative at \(x = a\).
\[ P_n''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3 \cdot 2c_4(x-a)^2 + 5 \cdot 4 \cdot 3 \cdot 2c_5(x-a)^3 + \ldots + n!c_n(x-a)^{n-2} \]
Setting \(P_n''(a) = f''(a)\) \[ \Rightarrow 2c_2 = f''(a) \Rightarrow c_2 = \frac{f''(a)}{2}. \]
   d. The third derivatives of \(P_3\) must equal \(f\) at \(x = a\). Setting
\[ P_n'''(a) = f'''(a) \Rightarrow 3 \cdot 2c_3 = f'''(a) \Rightarrow c_3 = \frac{f'''(a)}{3}. \]
   e. Continuing along in this manner,
\[ P_n^{(n)}(a) = f^{(n)}(a) \Rightarrow n!c_n = f^{(n)}(a) \Rightarrow c_n = \frac{f^{(n)}(a)}{n!}. \]
3. Using factorial notation, we can represent the \( n^{\text{th}} \) degree Taylor Polynomial of \( f \) centered at \( x = a \) as:

\[
P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

For example, the Third Degree Taylor Polynomial of \( f \) centered at \( x = a \) is

\[
P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} (x-a)^2 + \frac{f'''(a)}{3 \cdot 2} (x-a)^3.
\]

II. Taylor Polynomials

A. Formula

The \( n^{\text{th}} \) degree Taylor Polynomial of \( f \) centered at \( x = a \) is

\[
P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

B. Examples

1. Compute the 3\(^{rd} \) degree Taylor Polynomial of \( f \) centered at \( x = 2 \) for the function \( f(x) = \sqrt{2} + x \).

The following graph shows both \( f(x) = \sqrt{2} + x \) and \( P_3(x) \) on the interval \(-4 \leq x \leq 10\).

Note that the natural domain of \( f(x) = \sqrt{2} + x \) is \(-2 \leq x < \infty\) while the natural domain of \( P_3(x) \) is \(-\infty < x < \infty\).

Notice that the graphs are identical for approximately \( 0 < x < 4 \). We say the interval of convergence for \( f(x) \) and \( P_3(x) \) is \([0, 4]\).
2. Given \( f(x) = e^x \)
   a. Determine \( P_4(x) \), the fourth degree Taylor polynomial of \( f \) centered at \( x = 0 \).

   b. Use \( P_4(x) \) to estimate \( e^{0.1} \).

3. Given \( f(x) = \sin(x) \)
   a. Determine \( P_4(x) \), the fourth degree Taylor polynomial of \( f \) centered at \( x = \frac{\pi}{2} \).

   b. Determine \( P_5(x) \), the fifth degree Taylor polynomial of \( f \) centered at \( x = \frac{\pi}{2} \).

Note: For a particular choice of a function \( f \), \( x = a \), and integer \( n \), it may be the case that \( f^{(n)}(a) = 0 \). In that case, the \( n^{th} \) degree Taylor Polynomial of \( f \) centered at \( x = a \) would in fact be a polynomial of degree \( n - 1 \) or less.