Math 1205 Calculus/Sec. 4.3 Monotonic Functions and the First Derivative Test

I. Increasing/Decreasing Functions
   A. Definition: Let $f$ be a function defined on an interval $I$ and let $x_1$ and $x_2$ be any two points in $I$.
      1. $f$ increases on $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
      2. $f$ decreases on $I$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

   A function that is increasing or decreasing on $I$ is called monotonic on $I$.

   B. The First Derivative Test for Increasing/Decreasing.
      Suppose that $f$ is continuous on $[a, b]$ and differentiable on the open interval $(a, b)$.
      If $f'(x) > 0$ for all $x$ in $(a, b)$, then $f$ increases on $[a, b]$.
      If $f'(x) < 0$ for all $x$ in $(a, b)$, then $f$ decreases on $[a, b]$.

II. Local Extrema (Relative Extrema)
   A. Definitions
      1. A function $f$ has a local maximum value at an interior point $c$ of its domain if $f(x) \leq f(c)$ for all $x$ in some open interval containing $c$.
      2. A function $f$ has a local minimum value at an interior point $c$ of its domain if $f(x) \geq f(c)$ for all $x$ in some open interval containing $c$.
      3. An interior point of the domain of a function $f$ where $f'$ is zero or undefined is a critical point of $f$. (A stationary point exists where $f'(x) = 0$ and a singular point exists where $f'(x)$ is undefined.)

   B. The First Derivative Test for Local Extrema
      Suppose that $c$ is a critical point of a continuous function $f$ and that $f$ is differentiable at every point in some interval containing $c$ except possibly at $c$ itself.
      1. If $f'(x) > 0$ on $(a, c)$ and $f'(x) < 0$ on $(c, b)$ then $f$ has a local maximum of $f(c)$ at $x=c$.
      2. If $f'(x) < 0$ on $(a, c)$ and $f'(x) > 0$ on $(c, b)$ then $f$ has a local minimum of $f(c)$ at $x=c$.
      3. If $f'$ does not change signs at $x=c$, then $f$ has no local extrema at $x=c$. 
C. Steps in using the First Derivative Test for Local Extrema

1. Find \( f'(x) \)

2. Find the critical values. (Determine where \( f'(x) = 0 \) and/or \( f'(x) \) is undefined).

3. Determine the interval(s) where \( f \) is increasing (\( f'(x) > 0 \)) and interval(s) where \( f \) is decreasing (\( f'(x) < 0 \)).

4. a. If \( f \) is continuous, then \( f(c) \) is a relative maximum if \( f \) is increasing on \((a,c)\) followed by \( f \) decreasing on \((c,b)\).

   b. If \( f \) is continuous, then \( f(c) \) is a relative minimum if \( f \) is decreasing on \((a,c)\) followed by \( f \) increasing on \((c,b)\).

III. Examples

For the following functions, determine (a) where the function is increasing/decreasing and (b) the local extrema.

A. \( f(x) = x^2 \)

B. \( y = (2x - 1)^3 \)
C. \[ g(t) = \frac{1}{4}t^4 + \frac{1}{2}t^3 - 5t^2 \]

D. \[ h(x) = x - 2\sqrt{x} \quad \text{Note: domain is} [0, \infty) \]

E. \[ f(x) = xe^x \]