I. Rolle's Theorem
   A. Rolle’s Theorem
      Let $f$ be a function that satisfies the following three hypotheses:
         1. $f$ is continuous on the closed interval $[a,b]$.
         2. $f$ is differentiable on the open interval $(a,b)$.
         3. $f(a) = f(b)$
      Then there is a number $c$ in $(a,b)$ such that $f’(c) = 0$.

B. Geometric Interpretation
   Rolle’s Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one or it may have many.

C. Examples
   1. Verify that $f(x) = x \sqrt{x + 6}$ satisfies the three hypotheses of Rolle’s Theorem on the interval $[-6,0]$. Then find all numbers $c$ that satisfy the conclusion of Rolle’s Theorem.
2a. Construct a polynomial \( f(x) \) that has zeros at \( x = -2, -1, 0, 1, \) and 2.

b. Below are the graphs of \( f(x) \) and its derivative \( f'(x) \) together. How is what you see related to Rolle's Theorem.

c. Do \( g(x) = \sin x \) and its derivative \( g'(x) \) illustrate the same phenomenon?
3. Show that the equation \( x^3 + 4x + 3 \) has exactly one real solution.

II. The Mean Value Theorem

A. The Mean Value Theorem

Let \( f \) be a function that satisfies the following hypotheses:
1. \( f \) is continuous on the closed interval \([a,b]\).
2. \( f \) is differentiable on the open interval \((a,b)\).

Then there is a number \( c \) in \((a,b)\) such that
\[
 f'(c) = \frac{f(b) - f(a)}{b - a}
\]
or, equivalently
\[
 f(b) - f(a) = f'(c) (b - a).
\]

B. Physical Interpretation

1. The Mean Value Theorem says that at some interior point the instantaneous rate of change must equal the average rate of change.

2. There is an interior point where the slope of the tangent line must be equal to the slope of the secant line, i.e., the tangent line is parallel to the secant line.
C. Examples

1. The graph of \( f(x) = x + \frac{4}{x} \) is below. Graph the secant line through the points

2. \( f(x) = x + \frac{4}{x} \)

3. (1,5) and (8,8.5) on the graph below.

Find the number \( c \) that satisfies the conclusion of the Mean Value Theorem for this function \( f \) and the interval \([1,8]\). Then graph the tangent line at the pt \((c,f(c))\) and notice that it is parallel to the secant line.
2. Verify that \( f(x) = e^{-2x} \) satisfies the hypotheses of the Mean Value Theorem on the interval \([0,3]\). Then find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem.

III. Corollaries

A. Corollary 1: If \( f'(x) = 0 \) for all \( x \) in an open interval \((a,b)\), then \( f(x) = C \) for all \( x \in (a,b) \), where \( C \) is a constant.

**Proof:** If \( x_1 \) and \( x_2 \) are any two points in \((a,b)\), then \( f(x_1) = f(x_2) \), where \( x_1 < x_2 \).

Since \( f(x) \) satisfies the hypotheses of the Mean Value Thm, on \([x_1,x_2]\),

\[
f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

for some value \( c \) in \((x_1,x_2)\). Since \( f'(x) = 0 \) for all \( x \) in \((a,b)\)

\[
f'(c) = 0 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \Rightarrow f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1),
\]

B. Corollary 2: If \( f'(x) = g'(x) \) for all \( x \) in an open interval \((a,b)\), then there exists a constant \( C \) such that \( f(x) = g(x) + C \) for all \( x \) in \((a,b)\); i.e., \( f(x) - g(x) = C \) or \( f(x) - g(x) \) is a constant on \((a,b)\).

**Proof:** For all \( x \) in \((a,b)\), the derivative of the difference function \( h(x) = f(x) - g(x) \) is

\[
h'(x) = f'(x) - g'(x) = 0.
\]

Thus, \( h(x) = C \) on \((a,b)\) by Corollary 1. i.e., \( f(x) - g(x) = C \) on \((a,b)\), so \( f(x) = g(x) + C \).
IV. Finding Velocity and Position from Acceleration

A. Finding Velocity

Corollary 2 helps us find the velocity and displacement functions of a free falling body from rest with acceleration $9.8 \frac{m}{s^2}$. We know that the derivative of $v(t)$ is 9.8 and since $g(t) = 9.8t \Rightarrow g'(t) = 9.8$. By Corollary 2, $v(t) = 9.8t + C$ for some constant $C$. Since it is a free falling body, $v(0) = 0 \Rightarrow C = 0$. $\therefore v(t) = 9.8t$.

B. Finding Position

We know that the derivative of $s(t)$ is $9.8t$ and since $f(t) = 4.9t^2 \Rightarrow f'(t) = 9.8t$. By Corollary 2, $s(t) = 4.9t^2 + C$ for some constant $C$. If the initial height is $s(0) = h$, measured positive downward from the rest position, then $s(0) = 4.9(0)^2 + C = h \Rightarrow C = h$. $\therefore s(t) = 4.9t^2 + h$.

C. Example

Given $a(t) = -4 \sin(2t)$ with an initial velocity of $2 \frac{cm}{sec}$ and an initial position of $-3cm$, find the position function. Note: $v(0) = 2$, $s(0) = -3$