Math 1205 Calculus - Sec. 3.2 Differentiation Rules for Polynomials, Exponentials, Products and Quotients

I. Derivative Rules

A. Derivative of a Constant Function: If \( c \) is a constant, then \( \frac{d}{dx}(c) = 0 \).

1. Proof
\[
\frac{d}{dx}(c) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0
\]

2. Examples:
   a. \( \frac{d}{dx}(10) = 0 \)
   b. \( \frac{d}{dt}(-0.5) = 0 \)

B. Power Rule: If \( n \) is any real number, then \( \frac{d}{dx}(x^n) = n \cdot x^{n-1} \).

1. Proof if \( n \) is a positive integer
\[
\frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}
\]
\[
= \lim_{h \to 0} \frac{x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \binom{n}{3} x^{n-3}h^3 + \ldots + h^n - x^n}{h}
\]
\[
= \lim_{h \to 0} \frac{n \cdot x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \binom{n}{3} x^{n-3}h^3 + \ldots + h^n}{h}
\]
\[
= \lim_{h \to 0} \frac{h(n \cdot x^{n-1} + \binom{n}{2} x^{n-2}h + \binom{n}{3} x^{n-3}h^2 + \ldots + h^{n-1})}{h}
\]
\[
= n \cdot x^{n-1} \quad ; \quad n \in \text{positive integers}
\]

2. Examples:
   a. \( \frac{d}{dx}(x) = 1 \)
   b. \( \frac{d}{dx}(x^5) = 5x^4 \)
   c. \( \frac{d}{dp}(p^3) = 3p^2 \)
   d. \( \frac{d}{dx}(\sqrt{x^2}) = \frac{1}{2}x^{-\frac{1}{2}} \)
   e. For \( t \neq 0 \), \( \frac{d}{dt}\left(\frac{1}{t^2}\right) = -\frac{2}{t^3} \)

C. The Constant Multiple Rule: If \( f \) is a differentiable function of \( x \) and \( c \) is a constant, then
\( \frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}f(x) \)

Examples:
1. \( \frac{d}{dx}(7x) = 7 \)
2. \( \frac{d}{dx}(4x^5) = 20x^4 \)
3. \( \frac{d}{dy}(3y^2) = 6y \)
D. The Sum/Difference Rule: If \( f \) and \( g \) are differentiable functions of \( x \), then their sum \( f + g \) or their difference \( f - g \) is differentiable at every point where \( f \) and \( g \) are both differentiable. At such points \( \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \).

1. **Proof:**

\[
\begin{align*}
  k'(x) &= \lim_{h \to 0} \frac{k(x + h) - k(x)}{h} \\
  &= \lim_{h \to 0} \frac{f(x + h) + g(x + h) - (f(x) + g(x))}{h} \\
  &= \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right) \\
  &= \lim_{h \to 0} \left( \frac{f(x) - f(x)}{h} \right) + \lim_{h \to 0} \left( \frac{g(x + h) - g(x)}{h} \right) \\
  &= f'(x) + g'(x)
\end{align*}
\]

2. Examples:
   a. \( y = 3x^3 - \left( \frac{8}{x^4} \right) + 5 \)
   b. \( f(t) = 2t - \frac{3t^2}{4} + 18 - \pi^2 \)

E. Exponential Functions

1. **Definition of the Number \( e \)**
   a. Defn: \( e \) is the number such that \( \lim_{h \to 0} \left( \frac{e^h - 1}{h} \right) = 1 \)
   b. \( e \approx 2.7182818 \)

2. **Derivative of the Natural Exponential Function**

\[
\frac{d}{dx}(e^x) = e^x
\]

3. Examples
   a. \( f(x) = e^x + 4x - 3 \)
   b. \( g(x) = 5e^x - e^3 + 8 + \frac{2}{\sqrt{x}} \)
II. Piecewise Functions

A. Technique
   When determining if a piecewise function is differentiable at \( x=a \), you must check if
   (1) the is continuous at \( x=a \) and if (2) the one sided derivatives are equal.

B. Examples
   1. Let \( f(x) = \begin{cases} 
   8x^2 + 2 & \text{if } x \geq 1 \\
   16x - 6 & \text{if } x < 1
   \end{cases} \). Is \( f \) differentiable at \( x=1 \)? Explain your answer.

   2. Let \( h(x) = \begin{cases} 
   4x + 3 & \text{if } x < 5 \\
   x^2 - 2 & \text{if } x \geq 5
   \end{cases} \). Is \( h \) differentiable at \( x=5 \)? Explain your answer.

   3. Let \( g(x) = \begin{cases} 
   4x^3 - 6 & \text{if } x < 0 \\
   5x^2 + 3 & \text{if } x \geq 0
   \end{cases} \). Is \( g \) differentiable at \( x=0 \)? Explain your answer.
4. Use the definition of one-sided derivatives to show that \( g(x) = \begin{cases} 4x^3 - 6 & \text{if } x < 0 \\ 5x^2 + 3 & \text{if } x \geq 0 \end{cases} \) is not differentiable at \( x=0 \).

III. Application Examples

A. Find the equation of the tangent to the curve \( y = 3x^4 - 18x \) when \( x=2 \).

B. Find the equation of the tangent to the curve \( y = 7e^x + 2x \) when \( x=0 \).
C. Find where the tangent line for \( y = x^3 - x \) is parallel to the line \( x + y = 5 \).

D. At what point(s) does the graph of the curve \( f(x) = 4x^2 - 16x \) have a horizontal tangent line?

E. Def: A line is normal to a curve at a point if it is perpendicular to the curve’s tangent at that point. Recall that two lines are perpendicular if and only if the slope of one line is the negative reciprocal of the other.

Find the equation of the normal line to the graph of the curve of \( g(x) = 2x^2 + x + 4 \) when \( x = -3 \).

F. At what point(s) does the graph of the curve \( f(x) = \sqrt{x} \) have a vertical tangent line?

G. At what point(s) does the graph of the curve \( g(x) = \frac{1}{x} \) have a vertical tangent line?
IV. Derivative Rules (continued)

A. The Product Rule

1. **The Product Rule:** If \( f \) and \( g \) are both differentiable at \( x \), then so is their product \( fg \) and

\[
\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)].
\]

2. In words, the Product Rule says that the derivative of a product of 2 functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

3. Proof:

Let \( P(x) = f(x) \cdot g(x) \)

\[
P'(x) = \lim_{h \to 0} \left[ \frac{P(x+h) - P(x)}{h} \right] = \lim_{h \to 0} \left[ \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \right]
\]

Add the form of zero:

\[
f(x+h)g(x+h) - f(x)g(x) = \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} \cdot h + \frac{f(x+h)g(x) - f(x)g(x)}{h} \cdot h
\]

\[
P'(x) = \lim_{h \to 0} \left[ \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \right]
\]

\[
= \lim_{h \to 0} \left[ \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h} \right]
\]

\[
= \lim_{h \to 0} \left[ \frac{f(x+h)[g(x+h) - g(x)]}{h} + \frac{g(x)[f(x+h) - f(x)]}{h} \right]
\]

\[
= \lim_{h \to 0} f(x+h) \lim_{h \to 0} \left[ \frac{g(x+h) - g(x)}{h} \right] + \lim_{h \to 0} g(x) \lim_{h \to 0} \left[ \frac{f(x+h) - f(x)}{h} \right]
\]

\[
P'(x) = f(x)g'(x) + g(x)f'(x)
\]
4. Examples:
   a. Differentiate \( g(x) = (x^3 + 7)(x^2) \) two ways

   b. Find the derivative of \( f(x) = x^3 e^x \)

   c. \( \frac{d}{dx} \left( 9e^{\frac{x}{\sqrt[3]{x^3}}} \right) \)

   d. Find the tangent line to the curve of \( y = (x^2 - 8x + 1)(2x + 3) \) at \( x = 1 \).
B. The Quotient Rule

1. **The Quotient Rule**: If \( f \) and \( g \) are differentiable at \( x \) and \( g(x) \neq 0 \), then so is the quotient \( \frac{f}{g} \) is differentiable at \( x \) and 
\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}.
\]

2. In words, the Quotient Rule says that the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

3. Examples: Differentiate the following

   a. \( f(x) = \frac{2x + 1}{x^3 + 5x - 8} \)

   b. \( y = \frac{8e^t}{(2x^4 + 6x)} \)

   c. \( g(t) = \frac{-5t^3 + 8t + 9 + \sqrt{t}}{t} \)
d. Find the equation of the normal line to the graph of the curve of \( g(x) = \frac{2x}{x + 4} \) when \( x=4 \).

e. Determine the point(s) where the graph of \( y = \frac{e^x}{1 + x^2} \) has a horizontal tangent line.

C. Additional Examples

1. If \( f(x) = \sqrt{x} \cdot g(x) \), where \( g(9)=6 \) and \( g'(9)=-8 \), find \( f'(9) \).
2. Suppose that \( f(3) = 2, \ f'(3) = -7, \ g(3) = 10, \ g'(3) = 4 \). Find the values of

a. \( (f + g)'(3) \)

b. \( 7f'(3) \)

c. \( (fg)'(3) \)

d. \( \left(\frac{f}{g}\right)'(3) \)

V. Higher Order Derivatives

A. Definitions and Notation

1. Second Derivative

a. Defn: If \( f \) is a differentiable function, the second derivative is the derivative of the first derivative, i.e. \( \frac{d}{dx} \left( \frac{dy}{dx} \right) \). The second derivative is the rate of change of the first derivative wrt \( x \) or you can interpret the second derivative as the rate of change of the rate of change.

b. Notation: \( f''(x), \ y'', \ \frac{d^2y}{dx^2}, \ D^2y \)

2. Third Derivative

a. Defn: If \( f \) is a differentiable function, the third derivative is the derivative of the second derivative, i.e. \( \frac{d}{dx} \left( \frac{d^2y}{dx^2} \right) \). The third derivative is the rate of change of the second derivative wrt \( x \).

b. Notation: \( f'''(x), \ y''', \ \frac{d^3y}{dx^3}, \ D^3y \)
3. The $n^{th}$ Derivative

   a. The process of differentiation can be continued. The $n^{th}$ derivative is obtained by differentiating $f^{(n)}$ times.

   b. Notation of the $n^{th}$ derivative: $f^{(n)}(x)$, $y^{(n)}$, $\frac{d^n y}{dx^n}$, $D^n f(x)$.

B. Examples:
Find the indicated derivative

1. $y = 5x^2 + 19x - 2$; $\frac{d^3 y}{dx^3}$

*Note: The $(n+1)^{st}$ derivative of a polynomial of degree $n$ is __________.

2. $h(y) = 4y^5 - 4y^3 - 27$; $h^{(6)}(y)$

3. Find the fourth derivative of $g(t) = \frac{4}{t^2}$. 