I. Introduction

A. Secant Line

1. Definition: The secant line to the curve \( y = f(x) \) is the line connecting two points, \((a, f(a))\) and \((b, f(b))\), on the curve.

2. Slope of a secant line: 
\[
m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}
\]

3. Alternative form of the slope of a secant line:
\[
m_{\text{sec}} = \frac{f(a + h) - f(a)}{h} \text{ where } h = b - a.
\]

B. Average Rate of Change

1. The average rate of change of \( y = f(x) \) wrt \( x \) over the interval \([x_1, x_2]\) is
\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \text{ where } h = x_2 - x_1
\]
OR
\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a} = \frac{f(a + h) - f(a)}{h}, \text{ where } h = b - a
\]

2. \( \frac{f(x_1 + h) - f(x_1)}{h}, \text{ where } h = x_2 - x_1 \) is called the difference quotient of \( f \) at \( x_1 \).

3. The geometrical representation of the average rate of change is the slope of the secant line connecting the points \((x_1, f(x_1))\) and \((x_2, f(x_2))\).

4. The physics representation of the average rate of change is the average velocity over the time interval, \([t_1, t_2]\)
II. Tangent Line
A. What is a Tangent Line?
1. Wrt a Circle: (1) A line is tangent to a circle at a pt $P$ if the line $L$ passes through $P \perp$ to the radius at pt $P$. (2) Euclid states that a tangent is a line that intersects the circle once and only once.

2. Wrt a Line: The tangent is the line itself

3. Wrt a Curve: (1) “Draw the best circle inside of the curve” (circle of curvature), (2) a tangent line can touch the curve in more than one place, (3) Def: The tangent line to the curve $y = f(x)$ at the pt $P$ is the line $L$ though $P$ whose slope is the limit as pt $Q$ approaches pt $P$ of the slope of the secant line (from either side)

B. Calculating the Tangent Line
1. Using the graph: Draw the tangent line and then pick another point on the tangent and calculate this tangent using the point slope equation for a line. The main drawbacks for this method are the accuracy of your drawing and the grid lines on the graph paper.

1. Using the Function: Pick a second point $Q$ on the graph of the fn $y = f(x)$ close to pt $P$. Connect the points to form the secant line. As the change in $x$ gets very small (close to zero) the secant line will become the tangent line. The main drawback to this method is if the curve takes a drastic turn.

** Make a chart to determine the slope of the secant lines as pt $Q$ gets closer to pt $P$.**
Pt $Q$ approaching $P$ from the right.  (Secant line in red, tangent line in blue)

Pt $Q$ approaching $P$ from the left.

Pt $Q$ approaching $P$ from the right.  (Secant line in red, tangent line in blue)
C. Definitions
1. Def: The slope of the curve \( y = f(x) \) at the point \( P ((a, f(a)) \) is the number \( m \) given by:

\[
m = \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right), \quad \text{provided this limit exists.} \tag{1}
\]

OR

\[
m = \lim_{h \to 0} \left( \frac{f(a+h) - f(a)}{h} \right), \quad \text{provided this limit exists.} \tag{2}
\]

a. This is also called the slope of the tangent line.

b. A vertical tangent exists at \( x=a \) if the above limit goes to \( \pm \infty \) and the fn is defined at \( x=a \).

c. A horizontal tangent exists at \( x=a \) if the above limit is 0.

2. Def: The tangent line to the curve at the point point \( P ((a, f(a)) \), is the line through \( P \) with the slope \( m \) defined above.

D. Example
1. Consider the curve \( f(x) = x^2 + 1 \) at the point \( P (2,5) \).

   a. Find the slope of the tangent line to \( f(x) \) at pt \( P \) using def 1

   b. Find the slope of the tangent line to \( f(x) \) at pt \( P \) using def 2

   c. Find the tangent line to \( f(x) \) at pt \( P \) using the slope you found above.
2. Find the slope of the curve \( f(x) = \sqrt[3]{x} \) at \( x = 0 \).

III. (Instantaneous) Rate of Change
A. The instantaneous rate of change of \( y = f(x) \) with respect to \( x \) at \( x = x_i \) is

\[
\lim_{h \to 0} \left( \frac{f(x_i + h) - f(x_i)}{h} \right) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \text{ where } h = x_2 - x_1,
\]

provided the limit exists OR

\[
\lim_{h \to 0} \left( \frac{f(a + h) - f(a)}{h} \right), \text{ provided this limit exists.}
\]

B. The geometrical representation of the rate of change is the slope of the tangent line at the point \((a, f(a))\).

C. The physics representation of the rate of change is the velocity at \( t = a \).

Example: If a ball is thrown into the air with a velocity of 30 ft/s, its height (in feet) after \( t \) seconds is given by \( s = 30t - 16t^2 \). Find the velocity when \( t = 1.5 \) sec.

IV. Derivatives
A. Definition

Def\( ^2 \): The derivative of a function \( f \) at a number \( a \), denoted by \( f'(a) \), is

\[
f'(a) = \lim_{h \to 0} \left( \frac{f(a + h) - f(a)}{h} \right) \text{ if this limit exists.}
\]

Alternative form: \( f'(a) = \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right) \), provided this limit exists.
B. The derivative $f'(a)$ is the rate of change of $y = f(x)$ wrt $x$ when $x = a$.

C. Examples

1. Given $f(x) = \frac{1}{x}$, find $f'(3)$.

2. Given $f(x) = \sqrt{x - 3}$, find $f'(4)$.

3. Given $f(x) = x^2 - 3x$, find $f'(1)$.
4. The \( \lim \limits_{h \to 0} \frac{(3 + h)^2 - 9}{h} \) represents \( f'(a) \) for some function \( f(x) \) and some \( x=a \).
Determine \( f(x) \) and \( a \).

\[ f(x) = \text{___________} \quad a = \text{___________} \]

5. The \( \lim \limits_{h \to 0} \frac{\sin(\frac{3\pi}{2} + h) + 1}{h} \) represents \( f'(a) \) for some function \( f(x) \) and some \( x=a \).
Determine \( f(x) \) and \( a \).

\[ f(x) = \text{___________} \quad a = \text{___________} \]

6. The equation of the tangent to the fn \( f(x) \) at \( x=1 \) is \( y=16x-4 \). What are \( f'(1) \) and \( f(1) \)?

\[ f'(1) = \text{___________} \quad f(1) = \text{___________} \]

7. Janice is selling girl scouts cookies. Her total sales (in number of boxes) form a function, \( S(t) \), where \( t \) is time measured in days.

<table>
<thead>
<tr>
<th>( T ) (in days)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S(t) )</td>
<td>0</td>
<td>10</td>
<td>15</td>
<td>17</td>
<td>28</td>
<td>31</td>
</tr>
</tbody>
</table>

State the meaning of both \( S(5) = 31 \) and \( S'(5) = 6 \); include units for all numbers involved.
8. For the function below answer the following questions:

![Graph with labeled points A, B, C, D, E]

a) At what labeled point(s) is the derivative of the graph positive? _________
b) At what labeled point(s) is the derivative of the graph negative? _________
c) At what labeled point(s) is the derivative of the graph zero? ______________
d) At what labeled point does the graph have the greatest derivative? ________
e) At what labeled point does the graph have the least derivative? __________

V. Summary

A. The following statements all refer to the same thing.
   1. The derivative of \( f(x) \) at \( x = a \), \( f'(a) \)
   2. The rate of change of \( f(x) \) with respect to \( x \) at \( x = a \)
   3. The slope of the tangent to \( y = f(x) \) at \( x = a \)
   4. The slope of \( y = f(x) \) at \( x = a \)
   5. The limit of the difference quotient, \( \lim_{h \to 0} \left( \frac{f(a+h) - f(a)}{h} \right) \)
   6. Velocity of \( y = f(t) \) at \( t = a \)

B. The tangent line to \( y = f(x) \) at \( (a, f(a)) \) is the line through \( (a, f(a)) \) whose slope is equal to \( f'(a) \), the derivative of \( f \) at \( a \).