Math 1205 Calculus – Sec. 2.6: Continuity

I. Definitions
   A. **Def**: A function \( y = f(x) \) is continuous at an interior point \( x = c \) of its domain
      if \( \lim_{{x \to c}} f(x) = f(c) \).

   B. **Def**: A function \( y = f(x) \) is continuous at a left endpoint \( x = a \) of its domain
      if \( \lim_{{x \to a^-}} f(x) = f(a) \).

   C. **Def**: A function \( y = f(x) \) is continuous at a right endpoint \( x = b \) of its domain
      if \( \lim_{{x \to b^+}} f(x) = f(b) \).

II. Continuity Test
    A function \( f(x) \) is continuous at \( x = c \), if and only if:
    1. \( f(c) \) exists
    2. \( \lim_{{x \to c}} f(x) \) exists
    3. \( \lim_{{x \to c}} f(x) = f(c) \)

   **Note**: When dealing with endpoints we can replace #2 & #3 with left hand or right hand limits

III. Types of Discontinuities:
    A. Removable discontinuities:

   Fails step 3 of the continuity test
   \( \lim_{{x \to c}} f(x) \neq f(c) \)
   \( f(c) \) may or may not exist.

   Fails step 1 of the continuity test and the limit exists.
B. Jump Discontinuity

Fails step 2 of the continuity test because
\[ \lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x) . \]

\[ f(c) \text{ may or may not exist.} \]

C. Infinite Discontinuities

Fails step 2 of the continuity test because
\[ \lim_{x \to c^-} f(x) \to \pm \infty \]

\[ \text{or} \]

\[ \lim_{x \to c^+} f(x) \to \pm \infty . \]

\[ f(c) \text{ may or may not exist.} \]

Note: When classifying discontinuities, look at step 2.

D. Oscillating Discontinuity

Fails step 2 of the continuity test because
\[ \lim_{x \to c} f(x) \to DNE ; \text{ the fn oscillates too much to have a limit as } x \to c . \]

IV. Continuity on an Interval

A. Def: A function is continuous on an interval if it is continuous at every number in the interval. (If \( f \) is defined on only one side of an endpt of the interval, we understand continuous at the endpt to mean continuous from the right or continuous from the left.)
B. Examples

1. Continuous on a closed interval \([a, b]\)
   
   \(f(x) = \sqrt{9 - x^2}\) is continuous on its domain \([-3.3]\), i.e., \(f\) is continuous at all interior pts of \((-3, 3)\) and at the endpoints \(x = -3\) and \(x = 3\).

   ![Graph of \(f(x) = \sqrt{9 - x^2}\) on \([-3, 3]\)]

2. Continuous on an open interval \((a, b)\)
   
   ![Graph of \(f(x) = \sqrt{9 - x^2}\) on \((-3, 3)\)]

3. Continuous on an semi-open interval \([a, b)\)
   
   ![Graph of \(f(x) = \sqrt{9 - x^2}\) on \([-3, 3)\)]

D. Continuity on its domain

A **continuous function** is one that is continuous at every pt of its domain.

\(y = \sqrt{x}\) is continuous on its entire domain, \([0, \infty)\)

![Graph of \(y = \sqrt{x}\) on \([0, \infty)\)]
V. Theorems

A. Theorem 9 – Properties of Continuous Functions

If $f$ and $g$ are both continuous at $x=c$, $k$ is a constant and $r$ and $s$ are integers, then the following are continuous at $x=c$:

1. Sum/Difference: $f + g$ and $f - g$
2. Products: $f \cdot g$
3. Constant multiples: $k \cdot f$
4. Quotients: $\frac{f}{g}$, provided $g(c) \neq 0$
5. Powers: $f^r$, provided it is defined on an open interval containing $c$, where $r$ and $s$ are integers

B. Theorem 11 – Limits of Continuous Functions

If $f$ is continuous at $c$ and $\lim_{x \to a} g(x) = c$, then $\lim_{x \to a} f(g(x)) = f(c)$. In other words,

$$\lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) = f(c).$$

C. Theorem 10 – Composite of Continuous Functions

If $g$ is continuous at $c$, and $f$ is continuous at $g(c)$, then the composite fn $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at $x=c$.

D. Any polynomial is continuous everywhere; i.e., continuous on $\mathbb{R} = (-\infty, \infty)$

E. Any rational function is continuous wherever it is defined; i.e., continuous on its domain. (its denominator is not equal to zero).

F. Every root function, trigonometric function, inverse trigonometric function, exponential function, and logarithmic function is continuous on its domain.

G. The inverse fn of any continuous fn is continuous over its domain.
VI. Examples

A. Given the function below, discuss its continuity on the interval [0, 4].

\begin{align*}
x &= 0 \\
x &= 1 \\
x &= 2 \\
x &= 3 \\
x &= 4 \\
\end{align*}

Note: The fn is continuous except at the values found above.

B. State the intervals on which the function is continuous.

1. \( f(x) = \sqrt{x^2 - 4} \)
2. \( h(x) = \frac{x \cos(x^2)}{1 + x^4} \)
3. \( g(x) = \frac{x^2 + x}{x^3 - x} \)
4. \( y = \sec(x) \)
C. Given \( f(x) = \begin{cases} 
  x^3 & ; \quad x \leq -1 \\
  x & ; \quad -1 < x < 2 \\
  \frac{-x}{(x - 3)^2} & ; \quad 2 \leq x \leq 4 
\end{cases} \)

Determine where \( f(x) \) is discontinuous.

D. Find the value for \( c \) that will make \( f(x) = \begin{cases} 
  cx + 2 & ; \quad x < 2 \\
  cx^2 & ; \quad x \geq 2 
\end{cases} \) continuous on \((-\infty, \infty)\).
VII. Continuous Extension to a Point:
A. Process
Here we are basically explaining how we can deal with a removable point of discontinuity. Given a function $f(x)$ that has a removable point of discontinuity at $x=c$.

Then define $F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ \lim_{x \to c} f(x) = L & \text{if } x = c \end{cases}$

We now have a continuous function at all points.

B. Example
Find the discontinuities of the function $f(x) = \frac{4x - x^2}{2 - \sqrt{x}}$. If the function has a removable discontinuity, define a new function $F(x)$ that agrees with $f(x)$ everywhere except the discontinuity and is continuous at the discontinuity.

VIII. Intermediate Value Theorem for Continuous Functions:
A. Theorem 12
A fn $y = f(x)$ that is continuous on the closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if $y_0$ is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some number $c$ in $(a,b)$.
B. Example
Show that \( f(x) = \cos(x) - x \) has a root in the interval \([0, 1]\).