Non-Homogeneous Systems, Euler’s Method, and Exponential Matrix

We carry on nonhomogeneous first-order linear system of differential equations. We will show how Euler’s method generalizes to systems, giving us a numerical approach to solving systems. We then continue on the exponential matrix.

1. Variation of Parameters
2. Euler’s Method
3. Exponential Matrix
4. Diagonalization

1 Non-Homogeneous Systems

A nonhomogeneous first-order linear system of differential equations has the form \( y' = P(t)y + g(t) \), where \( g(t) \) is a nonzero matrix. \( P(t) \) is a square matrix, but the other symbols—\( y' \), \( y \), and \( g(t) \) denote column matrices. (This is in keeping with the notation introduced at the beginning of the chapter that column matrices would appear in **bold lowercase** letters, but square matrices would appear in *ITALIC CAPITAL* letters.

In our study of nonhomogeneous equations from previous chapters, we noted that the solution to a nonhomogeneous equation took the form \( y = y_C + y_P \), where \( y_C \) represented the *complementary* solution—the general solution to the related homogeneous equation, and \( y_P \) represented a *particular* solution to the nonhomogeneous part of the equation.
Example:

Consider the system of differential equations:

\[ \mathbf{y}' = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 3t + e^t - 1 \\ -2t - e^t - 2 \end{bmatrix}. \]

Show that \( \mathbf{y} = \begin{bmatrix} -\frac{3t}{5} - \frac{e^t}{4} + \frac{2}{25} \\ \frac{2t}{3} + \frac{e^t}{2} + \frac{8}{9} \end{bmatrix} \) is a solution to the system.

Find the general solution to the above system:

2 Solving Nonhomogeneous Equations

In sections §3.8 and §3.9, we discussed two techniques for solving nonhomogeneous higher-ordered equations: the Method of Undetermined Coefficients, and Variation of Parameters. Both techniques extend to nonhomogeneous systems, although the Method of Undetermined Coefficients is a much weaker method of solution in the case of homogeneous systems. (Example 1, p. 278, and the discussion immediately following this example point out the weaknesses in the method.) Therefore, we’ll focus our attention on the method of Variation of Parameters.
2.1 Variation of Parameters

We know that the complementary solution has the form $y_C = \Psi(t) \times c$, where $\Psi(t)$ denotes the solution matrix—a square matrix whose columns form a fundamental set—and $c$ denotes a column of constants.

In the Variation of Parameters method, we replace the constants $c_1, \ldots, c_n$, with functions $u_1, \ldots, u_n$, and solve the resulting equation. When we used this method in section §3.9, we discovered that it was easier to solve first for their derivatives, and then integrate; this continues to be true. However, solving first for the derivatives is complicated by the fact that we’ll have to find the inverse of the square matrix $\Psi(t)$ in the process.

Example 3

Using variation of parameters, solve the initial value problem:

$$y' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} y + \begin{bmatrix} e^{2t} \\ -2t \end{bmatrix}, \quad y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution:

First, we find the complementary solution to the homogeneous equation:

1. Find the eigenvalues: $\lambda = -1$ and $\lambda = 3$.
2. Find an eigenvector for each eigenvalue. When $\lambda = -1$, the eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and when $\lambda = 3$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
3. Write the complementary solution:

$$y_C(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

To find the particular solution to the nonhomogeneous equation:

1. Compute the inverse of the solution matrix:
2. Multiply the inverse of the solution matrix by the matrix \( g(t) \):

\[
\begin{bmatrix}
\frac{1}{2}e^t & \frac{1}{2}e^{-t} \\
\frac{1}{2}e^{-3t} & \frac{1}{2}e^{-3t}
\end{bmatrix}
\times
\begin{bmatrix}
e^{2t} \\
-2t
\end{bmatrix}
= \]

3. Find the antiderivative of Step 2:

\[
\int
\begin{bmatrix}
\frac{1}{2}e^{3t} + te^t \\
\frac{1}{2}e^{-t} - te^{-3t}
\end{bmatrix}
dt
= \begin{bmatrix}
e^{3t}/6 - e^{-t} + te^t \\
-e^{-t}/2 + e^{-3t} - te^{-3t}/3
\end{bmatrix}
\]

(We used integration by parts here.)

4. Multiply the complementary solution matrix \( \Psi(t) \) by the result of step 3:

\[
y_P(t) = \begin{bmatrix}
e^{-t} & e^{3t} \\
-e^{-t} & e^{3t}
\end{bmatrix}
\times
\begin{bmatrix}
e^{3t}/6 - e^t + te^t \\
-e^{-t}/2 + e^{-3t} - te^{-3t}/3
\end{bmatrix}
= \begin{bmatrix}
-e^{2t}/3 + 4t/3 - 8/9 \\
2e^{2t}/3 - 2t/3 + 10/9
\end{bmatrix}
\]

5. Combine the complementary solution with the particular solution:

\[
y(t) = y_C(t) + y_P(t) = \begin{bmatrix}
e^{-t} & e^{3t} \\
-e^{-t} & e^{3t}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
+ \begin{bmatrix}
-e^{2t}/3 + 4t/3 - 8/9 \\
2e^{2t}/3 - 2t/3 + 10/9
\end{bmatrix}
\]

6. Use the initial condition to solve for \( c_1 \) and \( c_2 \):
At time $t_0 = 0$,

$$y(0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{11}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system of equations, we get $c_1 = \frac{5}{6}$ and $c_2 = \frac{7}{18}$, so:

$$y(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix} \begin{bmatrix} \frac{5}{6} \\ \frac{7}{18} \end{bmatrix} + \begin{bmatrix} \frac{e^{2t}}{3} + \frac{4t}{3} - \frac{8}{9} \\ -\frac{2e^{2t}}{3} - \frac{2t}{3} + \frac{10}{9} \end{bmatrix}$$
Example 3: A shortcut

If we did not have an initial condition, steps 1–5 above would find the general solution. However, we can use the initial condition as a limit of integration in Step 3 to save us some work:

3’. Integrate the product over the interval \([t_0, t]\). Notice that since \(t\) is one of our endpoints, we’ll have to use a different variable of integration:

\[
\int_0^t \begin{bmatrix} \frac{1}{2}e^{3s} + se^s \\ \frac{1}{2}e^{-s} - se^{-3s} \end{bmatrix} ds = \begin{bmatrix} \frac{e^{3t}}{6} - e^t + te^t + \frac{5}{6} \\ e^{-t} + \frac{e^{-3t}}{9} + \frac{te^{-3t}}{3} + \frac{7}{18} \end{bmatrix}
\]

4’. Multiply the inverse of the solution matrix \(\Psi(t_0)\) (we’re evaluating the inverse at \(t_0\)) by the initial condition. Add this result to Step 3.

\[
\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} e^{3t} \\ -e^{-t} + \frac{e^{-3t}}{9} + \frac{te^{-3t}}{3} + \frac{7}{18} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}}{6} - e^t + te^t + \frac{5}{6} \\ -\frac{e^{-t}}{2} + \frac{e^{-3t}}{9} + \frac{te^{-3t}}{3} + \frac{7}{18} \end{bmatrix}
\]

5’. Multiply the complementary solution matrix \(\Psi(t)\) by the result of step 4. This completes the process – we don’t have to solve for \(c_1\) and \(c_2\).

\[
y(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix} \times \begin{bmatrix} \frac{e^{3t}}{6} - e^t + te^t + \frac{5}{6} \\ -\frac{e^{-t}}{2} + \frac{e^{-3t}}{9} + \frac{te^{-3t}}{3} + \frac{7}{18} \end{bmatrix} = \begin{bmatrix} \frac{7e^{3t}}{18} - \frac{e^{2t}}{2} + \frac{e^{-t}}{3} + \frac{4t}{3} - \frac{8}{9} \\ \frac{7}{18}e^{3t} - \frac{2e^{2t}}{3} - \frac{5e^{-t}}{6} - \frac{2t}{3} + \frac{10}{9} \end{bmatrix}
\]

(Check that this is the same solution we obtained on the previous page.)
Solve the initial value problem:

\[ y' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

3 Euler’s Method

Recall that Euler’s method gives an approximate solution to an initial value problem given in the form \( y' = f(t, y), \ y(t_0) = y_0 \). We choose a step size \( h \), and we will approximate the solution \( y \) at a series of equally spaced points \( t_1 = t_0 + h, \ t_2 = t_0 + 2h, \ldots, t_n = t_0 + nh \). We will use \( y_i \) to denote the approximation at \( t_i \), so \( y(t_i) \approx y_i \).

We get from the approximation at \( t_i \) to the approximation at \( t_{i+1} \) using the formula

\[ y_{i+1} = y_i + hf(t_i, y_i) \]

which uses the linearization of \( y \) to approximate the change in going from \( t_i \) to \( t_{i+1} \).

Example:

Suppose we wish to approximate the solution to the first order non-linear equation \( y' = \cos(y) + t, \ y(0) = -1 \), using a step size \( h = 0.1 \). We know that there exists a unique solution for all \( t \), since both \( f \) and \( f_y \) are continuous everywhere.

Thus we start with our initial condition, \( y(0) = -1 \). Then we can approximate

\[ y(t_1) = y(0.1) \approx y_1 = y(0) + hf(0, -1) = -1 + 0.1(\cos(-1) + 0) \approx -0.9460 \]

We can then use this to approximate \( y(0.2) \):

\[ y(0.2) \approx y_2 = -0.9460 + (0.1)(\cos(-0.9460) + 0.1) \approx -0.8775 \]

We can continue for as many steps as we need, approximating \( y(t_{i+1}) \) using \( t_i \) and \( y(t_i) \).

Euler’s method can be generalized easily to systems of first order equations. Euler’s method uses the linearization for \( y \) at each point \( t_i \) to predict the result at \( t_{i+1} \). If
we now consider a function $x$ (which is a vector valued function), we can still use a linearization at the point $t_i$ to predict the value of $x$ at $t_{i+1}$. We will still use the same formula, but it will now be based on the vector valued function $x$:

$$x(t_{i+1}) \approx x(t_i) + h x'(t_i)$$

Example:

Suppose we wish to solve the following system:

$$x'_1 = \frac{3}{2} - \frac{1}{10} x_1 + \frac{3}{40} x_2$$
$$x'_2 = 3 + \frac{1}{10} x_1 - \frac{1}{5} x_2$$

with initial conditions $x_1(0) = 25$ and $x_2(0) = 15$. (Although we will learn how to solve homogeneous systems shortly, we will not be studying how to find exact solutions to non-homogeneous systems such as this.) We will approximate the solution numerically using Euler’s method.

In this case, our initial condition for our vector valued function $x$ is

$$x(0) = \begin{pmatrix} 25 \\ 15 \end{pmatrix},$$

and we know that

$$x' = \begin{pmatrix} \frac{3}{2} - \frac{1}{10} x_1 + \frac{3}{40} x_2 \\ 3 + \frac{1}{10} x_1 - \frac{1}{5} x_2 \end{pmatrix}$$

Therefore, if we choose $h = 0.1$, we get the following approximations:

$$x(0.1) \approx \begin{pmatrix} 25 \\ 15 \end{pmatrix} + (0.1) \begin{pmatrix} \frac{3}{2} - \frac{1}{10} \cdot 25 + \frac{3}{40} \cdot 15 \\ 3 + \frac{1}{10} \cdot 25 - \frac{1}{5} \cdot 15 \end{pmatrix} = \begin{pmatrix} 25.0125 \\ 15.25 \end{pmatrix}$$

$$x(0.2) \approx \begin{pmatrix} 25.0125 \\ 15.25 \end{pmatrix} + (.1) \begin{pmatrix} \frac{3}{2} - \frac{1}{10} \cdot 25.0125 + \frac{3}{40} \cdot 15.25 \\ 3 + \frac{1}{10} \cdot 25.0125 - \frac{1}{5} \cdot 15.25 \end{pmatrix} \approx \begin{pmatrix} 25.03 \\ 15.50 \end{pmatrix}$$

If we continue this process, we get the following table of approximations for the values of $x_1$ and $x_2$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
<th>0.50</th>
<th>0.60</th>
<th>0.70</th>
<th>0.80</th>
<th>0.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>25.01</td>
<td>25.03</td>
<td>25.04</td>
<td>25.06</td>
<td>25.08</td>
<td>25.15</td>
<td>25.17</td>
<td>25.20</td>
<td>25.23</td>
</tr>
<tr>
<td>$x_2$</td>
<td>15.25</td>
<td>15.50</td>
<td>15.74</td>
<td>15.97</td>
<td>16.20</td>
<td>16.43</td>
<td>16.65</td>
<td>16.87</td>
<td>17.08</td>
</tr>
</tbody>
</table>
We can of course use this technique with a non-linear system as well:

**Example:**

Suppose we have the initial value problem given by the system

\[
\begin{align*}
    x_1' &= \sqrt{x_1} - x_2 + t \\
    x_2' &= 2x_1x_2 - 1
\end{align*}
\]

with initial conditions \(x_1(0) = 1, \ x_2(0) = 0\). Let’s use Euler’s method with step size \(h = .2\). We will denote the approximation of \(x_i\) at step \(j\) by \(x_{ij}\). So our initial conditions are \(x_{10} = 1\) and \(x_{20} = 0\), and \(x_{11}\) will be the approximation of \(x_1(t)\) at \(t_1 = 0.2\).

Thus, for our first step we get the following:

\[
x_{11} = x_{10} + (.2)(\sqrt{x_{10}} - x_{20} + t_0) = 1 + (.2)(\sqrt{1} - 0 + 0) = 1.2
\]

and

\[
x_{21} = x_{20} + (.2)(2x_{10}x_{20} - 1) = 0 + (.2)(2 \cdot 1 \cdot 0 - 1) = -.2
\]

Note that we needed **both** \(x_{10}\) and \(x_{20}\) to calculate the approximation at the next step for either function. We can then proceed to \(x_{12} \approx x_1(.4)\) and \(x_{22} \approx x_2(.4)\):

\[
x_{12} = x_{11} + (.2)(\sqrt{x_{11}} - x_{21} + t_1) = 1.2 + (.2)(\sqrt{1.2} - 0 + .2) \approx 1.4991
\]

while

\[
x_{22} = x_{21} + (.2)(2x_{11}x_{21} - 1) = -.2 + (.2) [(2)(1.2)(-.2) - 1] \approx -.4960
\]

We can continue as we need to, using our previous approximations for \(x_1\) and \(x_2\) to generate new approximations.

We can now of course also solve higher order differential equations using Euler’s method by first rewriting the equation as a system of first order equations.

**Example:**

Approximate the solution to \(y'' + y' = y + t\) with initial conditions \(y(0) = 0\) and \(y'(0) = 1\), using Euler’s method with a step size of \(h = .25\).

We begin by rewriting the equation as a system. We let \(u_1 = y\) and \(u_2 = y'\). Then we have first the equation

\[
u_1' = u_2
\]

9
and second the equation

\[ u_2' + u_2 = u_1 + t \quad \text{or} \quad u_2' = u_1 - u_2 + t \]

In matrix form our equation is

\[
\begin{pmatrix}
  u_1 \\
  u_2'
\end{pmatrix}
= \begin{pmatrix}
  0 & 1 \\
  1 & -1
\end{pmatrix}
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}
+ \begin{pmatrix}
  0 \\
  1
\end{pmatrix} t
\]

Our initial conditions are then \( u_1(0) = 0 \) and \( u_2(0) = 1 \), or

\[
\mathbf{u}_0 = \begin{pmatrix}
  0 \\
  1
\end{pmatrix}.
\]

Euler’s method becomes

\[
\mathbf{u}_{k+1} = \mathbf{u}_k + 0.25 \left[ \begin{pmatrix}
  0 & 1 \\
  1 & -1
\end{pmatrix} \mathbf{u}_k + \begin{pmatrix}
  0 \\
  t_k
\end{pmatrix} \right]
\]

So we set out to find \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \):

\[
\mathbf{u}_1 = \begin{pmatrix}
  0 \\
  1
\end{pmatrix} + 0.25 \left[ \begin{pmatrix}
  0 & 1 \\
  1 & -1
\end{pmatrix} \begin{pmatrix}
  0 \\
  0
\end{pmatrix} + \begin{pmatrix}
  0 \\
  1
\end{pmatrix} \right] = \begin{pmatrix}
  0 \\
  1
\end{pmatrix} + 0.25 \begin{pmatrix}
  1 \\
  -1
\end{pmatrix} = \begin{pmatrix}
  .25 \\
  .75
\end{pmatrix}
\]

\[
\mathbf{u}_2 = \begin{pmatrix}
  .25 \\
  .75
\end{pmatrix} + 0.25 \left[ \begin{pmatrix}
  0 & 1 \\
  1 & -1
\end{pmatrix} \begin{pmatrix}
  .25 \\
  .25
\end{pmatrix} + \begin{pmatrix}
  0 \\
  .25
\end{pmatrix} \right] = \begin{pmatrix}
  .25 \\
  .75
\end{pmatrix} + 0.25 \begin{pmatrix}
  0.7 \\
  -.25
\end{pmatrix} = \begin{pmatrix}
  .4375 \\
  .6875
\end{pmatrix}
\]

\[
\begin{align*}
u_{10} &= 0 \\
u_{20} &= 1 \quad \text{(initial conditions)}
\end{align*}
\]

We take the first step:

\[
\begin{align*}
u_{11} &= u_{10} + (.25)(u_{20}) = 0 + .25 = .25 \\
u_{21} &= u_{20} + (.25)(u_{10} - u_{20} + t_0) = 1 + (.25)(0 - 1 + 0) = .75
\end{align*}
\]

We then take a second step:

\[
\begin{align*}
u_{12} &= u_{11} + (.25)(u_{21}) = .25 + (.25)(.75) = .4375 \\
u_{22} &= u_{20} + (.25)(u_{11} - u_{21} + t_1) = .75 + (.25)(.25 - .75 + .25) = .6875
\end{align*}
\]
The actual solution to \( y'' + y' = y + t \) with initial conditions \( y(0) = 0 \) and \( y'(0) = 1 \) can be found by finding the complementary solution \( y_c(t) = c_1 \exp\left[\frac{(-1 + \sqrt{5})t}{2}\right] + c_2 \exp\left[\frac{(-1 - \sqrt{5})t}{2}\right] \), and then using the method of undetermined coefficients to find the non-homogeneous term. The solution is

\[
\frac{1 - \sqrt{5}}{2} \exp\left(\frac{-1 - \sqrt{5}}{2} t\right) + \frac{1 + \sqrt{5}}{2} \exp\left(\frac{-1 + \sqrt{5}}{2} t\right) - t - 1
\]

Evaluated at \( t = .5 \), we get .428693, so our approximation of \( u_{12} = .4375 \) is off by about .0088.

We also note that we happen to have an approximation \( u_{22} = .6875 \) for \( y'(.5) \). The actual value of \( y'(.5) \) is about .807381, which is not quite as good.

Of course, there are more sophisticated techniques than Euler’s method for estimating solutions (although they are based loosely on Euler’s method), but we will not study these more complicated techniques.

4 The Exponential Matrix

The exponential function \( y = e^{at} \) has turned up over and over in our study of differential equations. One of the reasons this function has been ubiquitous is that the function retains the same form when differentiating – the only thing that changes is the constant coefficient.

Is it possible to define a function on matrices that behaves the same way? In other words, is there some matrix-valued function \( f(A) \) whose derivative \( f'(A) = Af(A) \) is a multiple of itself?

The answer is yes, and we’ll use the exact same notation for this function:

Let \( A \) be a square matrix. We define the **exponential matrix** \( e^{At} \) to be the matrix whose derivative is \( A \) times itself.

4.0.1 What the exponential matrix is NOT:

Let \( A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \). Find \( e^{At} \). Unfortunately, the easy and obvious thing to try –

\[
\begin{bmatrix} e^{3t} & 1 \\ e^{-t} & e^{2t} \end{bmatrix}
\]
does not work.

We can check that it doesn’t work by taking the derivative: \[
\begin{bmatrix}
3e^{3t} & 0 \\
-e^{-t} & 2e^{2t}
\end{bmatrix}
\] and multiplying \( A \) times the matrix: \[
\begin{bmatrix}
3e^{3t} & 3 \\
-e^{3t} + 2e^{-t} & 2e^{2t} - 1
\end{bmatrix}
\]. These aren’t the same matrix, which means that we have to look for another method of finding the exponential matrix.

### 4.1 The Maclaurin series for \( e^{at} \)

Instead, we make use of the Maclaurin series for \( e^{at} \):

\[
e^{at} = \sum_{n=0}^{\infty} \frac{a^n t^n}{n!}.
\]

If we replace the scalar \( a \) with the matrix \( A \), we get:

\[
e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.
\]

In general, this summation is difficult to compute by hand, because of the tediousness of calculating repeated powers of \( A^n \), however, there are some shortcuts that we can use:

**Example 1**

A matrix is in **diagonal form** (or simply a **diagonal matrix**) if every entry not on the main diagonal of the matrix is zero.

For example, \[
\begin{bmatrix}
3 & 0 \\
0 & -5
\end{bmatrix}
\] is a diagonal matrix.
Let \( A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \) be a diagonal matrix. Find \( e^{At} \).

5 Diagonalization

In the last section, as one of the steps in performing the Method of Variations on a system, we multiplied by a matrix, and then later multiplied by the inverse of that matrix. A similar process, called **diagonalization** can be used to find the exponential of a matrix.

Example 2

Let \( A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \). Find \( e^{At} \).
Exercise 7

Let \( A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \). Find \( e^{At} \).

If matrix \( A \) has a full set of eigenvectors \( x_1, x_2, \ldots, x_n \), then \( A \) is diagonalizable:

\[
T^{-1}AT = D,
\]

where

\[
T = [x_1 \ x_2 \ \cdots \ x_n] \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}
\]

Note the positions of \( T \) and \( T^{-1} \): the eigenmatrix \( T \) appears after \( A \) and \( T^{-1} \) appears before.

Also note that: \( e^{At} = T e^{Dt} T^{-1}, \ e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \).

5.1 Decoupling transformation

Consider the linear system

\[
y' = Ay + g(t).
\]
If $A$ is diagonalizable (i.e. $T^{-1}AT = D$), then the above system can be rewritten as

$$T^{-1}y' = T^{-1}AT(T^{-1}y) + T^{-1}g(t).$$

Use the substitution $z = T^{-1}y$, we have a decoupled linear system

$$z' = Dz + \hat{g}(t),$$

where $\hat{g} = T^{-1}g$.

In the decoupled system, each row only concerns one unknown scalar and therefore can be solved as a first order linear differential equation:

$$z_1' = \lambda_1 z_1 + \hat{g}_1(t)$$
$$z_2' = \lambda_2 z_2 + \hat{g}_2(t)$$
$$\ldots$$

Finally, we can transform $z$ back to $y$ by

$$y = Tz$$

**Example:** Find the general solution of the following linear system by decoupling transformation.

$$y_1' = 3y_1 + 2y_2 + 4$$
$$y_2' = y_1 + 4y_2 + 1$$