SUPERCONVERGENCE OF DISCONTINUOUS FINITE ELEMENT SOLUTIONS FOR NONLINEAR HYPERBOLIC PROBLEMS

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Abstract. In this paper we study the superconvergence of the discontinuous Galerkin solutions for hyperbolic partial differential equations. On the first inflow element we prove that the $p$-degree discontinuous finite element solution converges at Radon points with an $O(h^{p+2})$ rate. We further show that the solution flux converges on average at $O(h^{2p+2})$ on element outflow boundary when no reaction terms are present. For reaction-convection problems we establish an $O(h^{\min\{2p+2, p+4\}})$ superconvergence rate of the flux on element outflow boundary. Globally, we prove that the flux converges at $O(h^{2p+1})$ for nonlinear conservation laws. Numerical computations indicate that our results extend to nonrectangular meshes and nonuniform polynomial degrees and that superconvergence occurs at Radon points on every element.

Key words. Discontinuous finite element methods, superconvergence, hyperbolic differential equations, conservation laws.

AMS subject classifications. 65M60, 65M25, 65M15.

1. Introduction. The discontinuous Galerkin (DG) finite element method has been used to solve first-order hyperbolic problems and is gaining in popularity. The DG method was first used for the neutron equation [18]. Since then, DG methods have been used to solve hyperbolic [6, 7, 8, 9, 12, 11, 13, 16], parabolic [14, 15], and elliptic [5, 4, 19] partial differential equations. For a more complete list of citations on the DG methods and its applications consult [10]. A main advantage of using discontinuous finite element basis is to simplify adaptive $p$– and $h$– refinement with hanging nodes.

Recently, Adjerid et al. [1] proved that smooth DG solutions of one-dimensional linear and nonlinear hyperbolic problems using $p$-degree polynomial approximations exhibit an $O(h^{p+2})$ superconvergence rate at the roots of Radon polynomial of degree $p + 1$. They used this result to construct asymptotically correct a posteriori error estimates. They further established a strong $O(h^{2p+1})$ superconvergence at the downwind end of every element. Krivodonova et al. [17] proved a superconvergence result on average on the outflow edge of every element of unstructured triangular meshes and constructed a posteriori error estimates that converge to the true error under mesh refinement. Adjerid and Massey [3] extended these results for multi-dimensional problems using rectangular meshes and presented an error analysis for linear problems and problems with a nonlinear reaction term. They showed that the leading term in the true local error is spanned by two $(p + 1)$-degree Radon polynomials in the $x$ and $y$ directions, respectively. They further showed that a $p$-degree discontinuous finite element solution exhibits an $O(h^{p+2})$ superconvergence at Radon points obtained as a tensor product of the roots of Radon polynomial of degree $p + 1$. For a linear model problem they established that, locally, the solution flux is $O(h^{2p+2})$ superconvergent.

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on average on the outflow element boundary and the global solution flux converges at an \( O(h^{2p+1}) \) rate on average at the outflow boundary of the domain. They used these superconvergence results to construct asymptotically exact \textit{a posteriori} error estimates for linear and nonlinear hyperbolic problems. In this paper, we extend the error analysis of Adjerid and Massey [3] to nonlinear hyperbolic scalar problems of the form

\[
\nabla \cdot \mathbf{F}(u) = h(x, y), \quad (x, y) \in \Omega = [0, 1]^2
\]

(1.1)

and

\[
\nabla \cdot \mathbf{F}(u) + \phi(u) = h(x, y), \quad (x, y) \in \Omega = [0, 1]^2.
\]

(1.2)

with boundary conditions

\[
u|_{\partial \Omega_{in}} = g.
\]

(1.3)

The inflow and outflow boundaries are defined as

\[
\partial \Omega_{in} = \{(x, y) \in \partial \Omega \mid \frac{\partial \mathbf{F}}{\partial u} \cdot \nu \leq 0\}
\]

(1.4a)

and

\[
\partial \Omega_{out} = \{(x, y) \in \partial \Omega \mid \frac{\partial \mathbf{F}}{\partial u} \cdot \nu > 0\}
\]

(1.4b)

where the boundary of \( \Omega \), \( \partial \Omega = \partial \Omega_{in} \cup \partial \Omega_{out} \) and \( \nu \) is the outward unit normal to \( \partial \Omega \).

In our analysis we assume \( \mathbf{F} : \mathbb{R} \to \mathbb{R}^2 \), \( \phi : \mathbb{R} \to \mathbb{R} \), \( u : \mathbb{R}^2 \to \mathbb{R} \), \( h \) and \( g \) to be analytic functions. We show that the DG solution of (1.1) is \( O(h^{p+2}) \) superconvergent at Radau points and the leading term in the error is a linear combination of two Radau polynomials. The flux is \( O(h^{2p+2}) \) superconvergent on average at the outflow boundary of the first inflow element and \( O(h^{2p+1}) \) superconvergent at the outflow boundary of the domain. For reaction problems (1.2) the flux is \( O(h^{\min(2p+2,p+4)}) \) superconvergent at the outflow boundary of the first inflow element. Thus, the error propagates at a higher order.

This paper is organized as follows: In §2 we state and prove the main superconvergence results. In §3 we show numerical results for several problems. We conclude with a few remarks.

2. Error analysis. In this section we will analyze the DG discretization error and show that the leading term in the error is proportional to \( p + 1 \)-degree Radau polynomials in the \( x \) and \( y \) directions. Prior to proving this result we need to recall a few preliminary lemmas.

The weak discontinuous Galerkin formulation is obtained by partitioning the domain \( \Omega \) into \( N = n \times n \) square elements and starting the integration with elements whose inflow boundary is on the domain inflow boundary.

In order to perform an error analysis we consider the first element \( \Delta = [0, h]^2 \) where \( h = 1/n \) and the space \( \mathcal{V}_p \) of polynomial functions such that

\[
\mathcal{P}_{p+1} \subset \mathcal{V}_p \cup \{x^{p+1}, y^{p+1}\}, \quad p \geq 0,
\]

(2.1a)
where $\mathcal{P}_k$ is the space of polynomials of degree $k$

$$\mathcal{P}_k = \{ q \mid q = \sum_{m=0}^{k} \sum_{i=0}^{m} c_i^m x^i y^{m-i}\}. \quad (2.1b)$$

These spaces are suboptimal but they lead to a very simple a posteriori error estimator. For efficiency reasons we consider the smallest spaces that satisfy (2.1)

$$\mathcal{V}_p = \{ V \mid V = \sum_{k=0}^{p} \sum_{i=0}^{k} c_i^k x^i y^{k-i} + \sum_{i=1}^{p} c_i^{p+1} x^i y^{p+1-i}\}. \quad (2.2)$$

We note that tensor product elements satisfy (2.1). Assuming $\frac{\partial E}{\partial x}(u(0,0)) = [\alpha_1, \alpha_2]\xi$, with $\alpha_i > 0$, $i = 1, 2$, one can prove that for $h$ small enough the inflow boundary of $\Delta$ is $\Gamma_{in} = \Gamma_1 \cup \Gamma_4$ where $\Gamma_1 = \{(x,0), 0 < x < h\}$ and $\Gamma_4 = \{(0,y), 0 < y < h\}$. The outflow boundary $\Gamma_{out} = \Gamma_2 \cup \Gamma_3$ with $\Gamma_2 = \{(h,y), 0 < y < h\}$ and $\Gamma_3 = \{(x,h), 0 < x < h\}$. To simplify the global error analysis, we also assume that each component of $\frac{\partial E}{\partial x}(u(x,y))$ is positive on $\Omega$.

The discontinuous Galerkin method for (1.1) consists of determining $U(x,y) \in \mathcal{V}_p$ on $\Delta$ such that

$$\int_{\Gamma_{in}} \nu \cdot \left( \mathbf{F}(U^-) - \mathbf{F}(U) \right) Vd\sigma + \int_{\Delta} \left[ \nabla \cdot \mathbf{F}(U) - h(x,y) \right] Vdxdy = 0,$$

$$\forall \ V \in \mathcal{V}_p. \quad (2.3)$$

The boundary data $U^-$ on $\Gamma_{in}$ is

$$U^-(x,y) = \begin{cases} \pi g, & \text{if } (x,y) \in \Gamma_1, \\ \pi g, & \text{if } (x,y) \in \Gamma_4, \end{cases}, \quad (2.4)$$

where $\pi w$ is the $p$-degree polynomial that interpolates $w$ at the roots of $(p+1)$-degree right Radau polynomial

$$R^+_{p+1}(\xi) = L_{p+1}(\xi) - L_p(\xi), \quad -1 \leq \xi \leq 1, \quad (2.5)$$

with $L_p$ being Legendre polynomial of degree $p$.

Once we determine the solution on the first element $\Delta$ we proceed to the elements whose inflow boundaries are either on the inflow boundary of $\Omega$ or an outflow boundary of $\Delta$ and continue this process until the solution is determined in the whole domain. On an element whose inflow boundary is not on the boundary of $\Omega$, $U^-$ is defined as

$$U^-(x,y) = \lim_{s \to -0} U((x,y) + s\nu), \quad (x,y) \in \Gamma_{in}. \quad (2.6)$$

The discontinuous Galerkin solution satisfies the DG orthogonality condition which is obtained by multiplying (1.1) by $V \in \mathcal{V}_p$, integrating over the element $\Delta$ and applying Green’s formula to obtain

$$\int_{\Gamma} \nu \cdot \mathbf{F}(u) Vd\sigma - \int_{\Delta} \mathbf{F}(u) \cdot \nabla Vdxdy = \int_{\Delta} h(x,y)Vdxdy. \quad (2.7)$$
Applying Green’s formula to (2.3) yields
\[
\int_{\Gamma_{in}} \mathbf{\nu} \cdot \mathbf{F}(U^-) V d\sigma + \int_{\Gamma_{out}} \mathbf{\nu} \cdot \mathbf{F}(U) V d\sigma - \int \int_{\Delta} \mathbf{F}(U) \cdot \nabla V dxdy
\]
\[
= \int \int_{\Delta} h(x, y) V dxdy. \tag{2.8}
\]
Subtracting (2.7) from (2.8) we obtain the DG orthogonality condition
\[
\int_{\Gamma_{in}} \mathbf{\nu} \cdot (\mathbf{F}(U^-) - \mathbf{F}(u)) V d\sigma + \int_{\Gamma_{out}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) V d\sigma
\]
\[
- \int \int_{\Delta} (\mathbf{F}(U) - \mathbf{F}(u)) \cdot \nabla V dxdy = 0, \quad \forall V \in \mathcal{V}_p. \tag{2.9}
\]
Using the mapping of \( \Delta = [0, h]^2 \) into the canonical element \( \hat{\Delta} = [-1, 1]^2 \) defined by \( x = h(1 + \xi)/2 \) and \( y = h(1 + \eta)/2 \) and \( \hat{u}(\xi, \eta) = u(x(\xi), y(\eta)) \) we obtain the DG orthogonality condition (2.9) on the canonical element
\[
\int_{\Gamma_{in}} \mathbf{\nu} \cdot (\mathbf{\hat{F}}(\hat{U}^-) - \mathbf{\hat{F}}(\hat{u})) \hat{V} d\hat{\sigma} + \int_{\Gamma_{out}} \mathbf{\nu} \cdot (\mathbf{\hat{F}}(\hat{U}) - \mathbf{\hat{F}}(\hat{u})) \hat{V} d\hat{\sigma}
\]
\[
- \int \int_{\hat{\Delta}} (\mathbf{\hat{F}}(\hat{U}) - \mathbf{\hat{F}}(\hat{u})) \cdot \nabla \hat{V} d\xi d\eta = 0, \quad \forall \hat{V} \in \hat{\mathcal{V}}_p. \tag{2.10}
\]
In the remainder of this paper we omit the \( \hat{\cdot} \) unless we feel it is needed for clarity.

Now, we recall the following two preliminary lemmas.

**Lemma 2.1.** If \( Q_k \in \mathcal{V}_k \) and \( \alpha \in \mathbb{R}^2 \) satisfy
\[
\int_{\Gamma_{out}} \alpha \cdot \mathbf{\nu} Q_k V d\sigma - \int \int_{\Delta} \alpha \cdot \nabla V Q_k d\xi d\eta = 0, \quad \forall V \in \mathcal{V}_p, \ k \leq p, \tag{2.11}
\]
then
\[
Q_k = 0, \quad k \leq p. \tag{2.12}
\]

**Proof.** See Adjerid and Massey [3].

**Lemma 2.2.** Let \( w \in C^\infty(0, h) \) and \( \pi w \) be a p-degree polynomial that interpolates \( w \) at Radau points on \([0, h]\). Then the interpolation error
\[
w(x(\xi)) - \pi w(x(\xi)) = \sum_{k=p+1}^{\infty} Q_{k+1}(\xi) h^k, \tag{2.13a}
\]
where
\[
Q_{p+1}(\xi) = \frac{w^{(p+1)}(0)}{2^{p+1}(p+1)!} (\xi - \xi_0)(\xi - \xi_1) \cdots (\xi - \xi_p) = c_{p+1} R_{p+1}(\xi), \tag{2.13b}
\]
\[ Q_k^-(\xi) = R_{p+1}^+\xi_{p+1}(\xi), \quad k > p + 1, \quad (2.13c) \]

with \( r_k(\xi) \) being a polynomial of degree \( k \).

**Proof.** See Adjerid and Massey [3]. \( \Box \)

Now we are ready to state the main result for nonlinear conservation laws.

**Theorem 2.3.** Let \( u \) and \( U \) be the solution of (1.1) and (2.3), respectively. Then the local finite element error

\[ \epsilon = U - u \quad (2.14) \]

can be written as

\[ \epsilon(\xi, \eta) = \sum_{k=p+1}^{\infty} h^k Q_k(\xi, \eta), \quad (2.15) \]

where

\[ Q_{p+1} = \beta_1 R_{p+1}^+(\xi) + \beta_2 R_{p+1}^+(\eta). \quad (2.16) \]

Furthermore, at the outflow boundary of the physical element \( \Delta \)

\[ \int_{\Gamma_{\text{out}}} \mathbf{v} \cdot (\mathbf{F}(u) - \mathbf{F}(U)) \, d\sigma = O(h^{2p+2}). \quad (2.17) \]

The global error satisfies

\[ \int_{\partial\Omega_{\text{out}}} \mathbf{v} \cdot (\mathbf{F}(u) - \mathbf{F}(U)) \, d\sigma = O(h^{2p+1}). \quad (2.18) \]

**Proof.** The proof is established using the DG orthogonality condition (2.10).

First we write the Taylor series of \( \mathbf{F} \) about \( u \) to obtain

\[ \mathbf{F}(U) - \mathbf{F}(u) = \sum_{k=1}^{\infty} \frac{\mathbf{F}^{(k)}(u)}{k!} (U - u)^k. \quad (2.19) \]

Assuming \( U^- \) to be the interpolant of \( u \) on the inflow boundaries described in (2.4) and using (2.13) we see that on an inflow boundary edge

\[ \mathbf{F}(U^-) - \mathbf{F}(u) = \mathbf{F}^{(1)}(u)(U^- - u) + O(h^{2p+2}). \quad (2.20) \]

The Maclaurin series of \( \mathbf{F}^{(1)}(u) \) with respect to \( h \) can be written as

\[ \mathbf{F}^{(1)}(u) = \sum_{l=0}^{\infty} \phi_{l}^{[1]} h^l, \quad (2.21a) \]

where

\[ \phi_{l}^{[1]} = \frac{1}{l!} \frac{d^l \mathbf{F}^{(1)}(u(x,y))}{dh^l} \Big|_{h=0}. \quad (2.21b) \]
Combining (2.13), (2.21) and (2.20) to obtain

\[
\nu \cdot (F(U) - F(u)) = \begin{cases} 
  h^{p+1}R_{p+1}^1(\xi) [\sum_{k=0}^p h^k r_{1,k}(\xi)] + O(h^{2p+2}) & \text{on } \Gamma_1 \\
  h^{p+1}R_{p+1}^1(\eta) [\sum_{k=0}^p h^k r_{4,k}(\eta)] + O(h^{2p+2}) & \text{on } \Gamma_4,
\end{cases}
\]

(2.22)

where \(r_{1,k}, r_{4,k} \in P_k\).

The Maclaurin series of \(U - u\) and \(F^{(k)}(u)\) with respect to \(h\) can be written as

\[
U - u = \sum_{l=0}^{\infty} Q_l h^l,
\]

(2.23a)

where

\[
Q_l(\xi, \eta) = \frac{1}{l!} \left. \frac{d^l (U - u)}{dh^l} \right|_{h=0}.
\]

(2.23b)

We also have

\[
F^{(k)}(u) = \sum_{l=0}^{\infty} \Phi^{[l]}_l h^l,
\]

(2.24a)

where

\[
\Phi^{[l]}_l(\xi, \eta) = \frac{1}{l!} \left. \frac{d^l F^{(k)}(u)}{dh^l} \right|_{h=0}.
\]

(2.24b)

Combining (2.19), (2.23) and (2.24) yields

\[
F(U) - F(u) = \sum_{k=0}^{\infty} W_k h^k,
\]

(2.25)

where \(W_k \in P_k \times P_k\).

Substituting (2.22) and (2.25) in (2.10) and collecting terms having the same powers of \(h\) lead to

\[
\sum_{k=0}^{P} h^k \left( \int_{\Gamma_{int}} \nu \cdot W_k \, V \, d\sigma - \int_{\Gamma} W_k \cdot \nabla V \xi d\eta \right) - \nabla V \xi d\eta
\]

\[
= \sum_{k=p+1}^{\infty} h^k \left( \int_{\Gamma_{int}} \nu \cdot W_k \, V \, d\sigma + \int_{\Gamma_{int}} \nu \cdot W_k \, V \, d\sigma - \int_{\Gamma} \nu \cdot W_k \, V \, d\sigma - \nabla V \xi d\eta \right) = 0, \quad \forall V \in V_p,
\]

(2.26)

where using (2.22) we have

\[
\nu \cdot W_k = \begin{cases} 
  R_{p+1}^1(\xi)r_{1,k-p-1}(\xi) & \text{on } \Gamma_1 \\
  R_{p+1}^1(\eta)r_{4,k-p-1}(\eta) & \text{on } \Gamma_4
\end{cases}, \quad p + 1 \leq k \leq 2p + 1.
\]

(2.27)
The $O(1)$ term in (2.26) with $V = 1$ yields

$$W_0 = Q_0 \sum_{l=0}^{\infty} \frac{\Phi_0^{(l+1)} Q_0^l}{(l + 1)!} = 0,$$  

(2.28)

which in turn leads to $Q_0 = 0$. We note that, for instance, for $F(u) = [u^2 / 2, u]^T$ there exists $Q_0 \neq 0$ solution of (2.28) which corresponds to a non physical DG solution with $\epsilon = O(1)$. Here, we will not consider such non physical solutions.

By induction the $O(h^k)$, $k < p + 1$ leads to

$$\int_{\Gamma_{\text{out}}} \nu \cdot \Phi_0^{[1]} Q_k V d\sigma - \int_{\Delta} Q_k \Phi_0^{[1]} \cdot \nabla V d\xi d\eta = 0, \forall V \in \mathcal{V}_p.$$  

(2.29)

Applying Lemma 2.1 we establish that $Q_k = 0$, $k = 1, 2, \ldots, p$.

Following the same line of reasoning as in [3], we show that the leading term $Q_{p+1}$ can be split as

$$Q_{p+1} = \frac{1}{(p + 1)!} \frac{d^{p+1}(U - u)}{dh^{p+1}}(\xi, \eta)|_{h=0} = \tilde{Q}_{p+1} + \tilde{Q}_p,$$  

(2.30a)

where $\tilde{Q}_p(\xi, \eta) \in \mathcal{V}_p$ and

$$\tilde{Q}_{p+1} = c_{p+1} \frac{1}{2^{p+1}(p + 1)!} \frac{\partial^{p+1} u}{\partial x^{p+1}(0, 0) R_{p+1}^+}(\xi)$$

$$+ c_{p+1} \frac{1}{2^{p+1}(p + 1)!} \frac{\partial^{p+1} u}{\partial y^{p+1}(0, 0) R_{p+1}^+}(\eta).$$  

(2.30b)

Substituting (2.30) in the $O(h^{p+1})$ term of the series (2.26) leads to

$$\int_{\Gamma_{\text{in}}} \Phi_0^{[1]} \cdot \nu Q_{p+1} V d\sigma + \int_{\Gamma_{\text{out}}} \Phi_0^{[1]} \cdot \nu Q_{p+1} V d\sigma - \int_{\Delta} \Phi_0^{[1]} \cdot \nabla V Q_{p+1} d\xi d\eta$$

$$+ \int_{\Gamma_{\text{out}}} \Phi_0^{[1]} \cdot \nabla \tilde{Q}_{p+1} V d\sigma = \int_{\Delta} \Phi_0^{[1]} \cdot \nabla \tilde{Q}_{p+1} d\xi d\eta = 0, \forall V \in \mathcal{V}_p.$$  

(2.31)

Using (2.13) and (2.30b) we can show that

$$\int_{\Gamma_{\text{in}}} \Phi_0^{[1]} \cdot \nu Q_{p+1} V d\sigma + \int_{\Gamma_{\text{out}}} \Phi_0^{[1]} \cdot \nu Q_{p+1} V d\sigma - \int_{\Delta} \Phi_0^{[1]} \cdot \nabla V Q_{p+1} d\xi d\eta = 0,$$

$$\forall V \in \mathcal{V}_p.$$  

(2.32)

Combining (2.31) and (2.32) with Lemma 2.1 leads to $\tilde{Q}_p = 0$. Using (2.30) we establish (2.16).

Using (2.13) we can show that

$$\int_{\Gamma_{\text{in}}} \Phi_0^{[1]} \cdot \nu Q_{p+1} V d\sigma = 0, \forall V \in \mathcal{V}_{2p-k}, k = p + 1, \ldots, 2p.$$  

(2.33)
Using (2.33), the $O(h^k)$, $p + 1 \leq k \leq 2p$, term of (2.26) yields

\[ \int_{\Gamma_{\text{out}}} \Phi^1 \cdot \mathbf{\nu} Q_k V d\sigma - \int_{\Delta} \Phi^1 \cdot \nabla V Q_k \, d\xi \, d\eta = 0, \quad \forall V \in \mathcal{V}_{2p-k}. \]  

(2.34)

Testing against $V = 1$ we obtain

\[ \int_{\Gamma_{\text{out}}} \Phi^1 \cdot \mathbf{\nu} Q_k \, d\sigma = 0, \quad k = p + 1, \ldots, 2p, \]  

(2.35)

which establishes (2.17).

Next we prove global superconvergence by showing that on every element $\Delta$

\[ \int_{\Gamma_{\text{in}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) \, d\sigma + \int_{\Gamma_{\text{out}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) \, d\sigma = 0. \]  

(2.36)

Summing over all elements we obtain

\[ \int_{\partial \Omega_{\text{in}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) \, d\sigma + \int_{\partial \Omega_{\text{out}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) \, d\sigma = 0. \]  

(2.37)

Using (2.13) leads to (2.18). \( \square \)

Next, we will describe similar results for problems of the form (1.2) where the DG weak formulation consists of determining $U(x, y) \in \mathcal{V}_p$ on $\Delta$ such that

\[ \int_{\Gamma_{\text{in}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) \, V \, d\sigma + \int_{\Delta} [\nabla \cdot \mathbf{F}(U) + \phi(U) - h(x, y)] V \, dx \, dy = 0, \]  

\[ \forall \, V \in \mathcal{V}_p. \]  

(2.38)

In the following theorem we state a superconvergence result for nonlinear hyperbolic problem with reaction terms.

**Theorem 2.4.** Let $u$ and $U$ be the solution of (1.2) and (2.38), respectively. If $u$, $\mathbf{F}$ and $\phi$ are analytic functions, then the local error estimates (2.15) and (2.16) hold. Furthermore, we have the following superconvergence results on the first inflow element

\[ \int_{\Gamma_{\text{in}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) = O(h^{\min(2p+2, p+4)}) \]  

(2.39)

and

\[ \int_{\Gamma_{\text{out}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) \, d\sigma + \int_{\Delta} [\phi(U) - \phi(u)] \, dx \, dy = O(h^{2p+2}). \]  

(2.40)

Globally, we have the strong superconvergence

\[ \int_{\partial \Omega_{\text{in}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) \, d\sigma + \int_{\Omega} [\phi(U) - \phi(u)] \, dx \, dy = O(h^{2p+1}). \]  

(2.41)

**Proof.** The DG orthogonality condition is

\[ \int_{\Gamma_{\text{in}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) \, V \, d\sigma + \int_{\Gamma_{\text{out}}} \mathbf{\nu} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) \, V \, d\sigma - \]
\[
\int \int_\Delta (\mathbf{F}(U) - \mathbf{F}(u)) \cdot \nabla V + [\phi(u) - \phi(U)] V dxdy = 0, \ \forall V \in \mathcal{V}_p. \tag{2.42}
\]

On the canonical element $[-1,1]^2$ (2.42) becomes

\[
\int_{\Gamma_{in}} \nu \cdot (\mathbf{F}(U^-) - \mathbf{F}(u)) V d\sigma + \int_{\Gamma_{out}} \nu \cdot (\mathbf{F}(U) - \mathbf{F}(u)) V d\sigma
\]

\[
-\int \int_\Delta (\mathbf{F}(U) - \mathbf{F}(u)) \cdot \nabla V + \frac{h}{2} [\phi(u) - \phi(U)] V d\xi d\eta = 0, \ \forall V \in \mathcal{V}_p. \tag{2.43}
\]

Applying Taylor series to $\phi$ about $u$ we have

\[
\phi(u) - \phi(U) = -a(u)e - \frac{\epsilon^2}{2} \phi''(\bar{u}), \ a(u) = \phi'(u). \tag{2.44a}
\]

The Maclaurin series of $a(u)$ with respect to $h$ yields

\[
a(u) = 2 \sum_{k=0}^\infty h^k \tilde{Q}_k(\xi, \eta), \quad \tilde{Q}_k(\xi, \eta) = \frac{1}{2} \frac{\phi^{(k+1)}(u(x(\xi), y(\eta)))}{k!} \frac{d^k u}{d(\xi, \eta)^k} \bigg|_{(\xi, \eta) = 0} \in \mathcal{P}_k. \tag{2.44b}
\]

Now we substitute (2.44), (2.22) and (2.25) in (2.43), collect terms having the same powers of $h$ to obtain

\[
\left( \int_{\Gamma_{out}} \nu \cdot \mathbf{W}_0 V d\sigma - \int \int_\Delta \mathbf{W}_0 \cdot \nabla V d\xi d\eta \right) +
\]

\[
\sum_{k=1}^p h^k \left( \int_{\Gamma_{out}} \nu \cdot \mathbf{W}_k V d\sigma - \int \int_\Delta [\mathbf{W}_k \cdot \nabla V - Z_{k-1} V] d\xi d\eta \right) +
\]

\[
\sum_{k=p+1}^\infty h^k \left( \int_{\Gamma_{in}} \nu \cdot \mathbf{W}_k^- V d\sigma + \int_{\Gamma_{out}} \nu \cdot \mathbf{W}_k V d\sigma - \int \int_\Delta [\mathbf{W}_k \cdot \nabla V - Z_{k-1} V] d\xi d\eta \right) = 0, \ \forall V \in \mathcal{V}_p, \tag{2.45}
\]

where

\[
Z_k = \sum_{l=0}^k \bar{Q}_l Q_{k-l} \tag{2.46}
\]

and $\nu \cdot \mathbf{W}_k^-$ is given in (2.27).

Following the same line of reasoning as in Theorem 2.3 we prove (2.15) and (2.16) for problems with a nonlinear reaction term. We note that the term in (2.44a) involving $\epsilon^2$ is higher order and does not contribute to our leading terms and that

\[
Z_m = \begin{cases} 0 & \text{if } m \leq p \\ \sum_{l=0}^{m-p-1} \bar{Q}_l Q_{m-l} & \text{if } m \geq p+1 \end{cases}. \tag{2.47}
\]
We prove the strong superconvergence result (2.39) for nonlinear hyperbolic problems with reaction terms by setting $V = 1$ in (2.45). Using (2.13), (2.15) and (2.16) to obtain

$$\int_{G_{in}} \mathbf{v} \cdot \Phi_{0}^{[1]} Q_{p+1} d\sigma = 0. \tag{2.48}$$

Setting $V = 1$ in the $O(h^k)$, $k > p + 1$ in (2.45) leads to

$$\int_{G_{in}} \mathbf{v} \cdot \Phi_{0}^{[1]} Q_{k} d\sigma + \int_{G_{out}} \mathbf{v} \cdot \Phi_{0}^{[1]} d\sigma + \int_{\Delta} Z_{k-1} d\xi d\eta = 0. \tag{2.49}$$

Using (2.22) and (2.47) the $O(h^{p+2})$ term leads to

$$\int_{G_{out}} \mathbf{v} \cdot \Phi_{0}^{[1]} Q_{p+2} d\sigma = 0. \tag{2.50}$$

The $O(h^{p+3})$ term yields

$$\int_{G_{out}} \mathbf{v} \cdot \Phi_{0}^{[1]} Q_{p+3} d\sigma = - \int_{\Delta} \tilde{Q}_{0} Q_{p+2} d\xi d\eta, \tag{2.51}$$

which is not necessarily zero. Thus, we establish (2.39).

Now, using (2.42) with $V = 1$ and (2.22) establishes (2.40).

Letting $V = 1$ in (2.42) and summing over all elements lead to

$$\int_{\partial\Omega_{n}} \mathbf{v} \cdot (\mathbf{F}(\mathbf{U}^{-}) - \mathbf{F}(\mathbf{u})) d\sigma + \int_{\partial\Omega_{out}} \mathbf{v} \cdot (\mathbf{F}(\mathbf{U}) - \mathbf{F}(\mathbf{u})) d\sigma +$$

$$\int \int_{\Omega} [\phi(\mathbf{U}) - \phi(\mathbf{u})] d\mathbf{x} d\mathbf{y} = 0. \tag{2.52}$$

Applying (2.13) and (2.22) yields (2.41) which completes the proof of Theorem 2.4.

3. Numerical Examples. We will consider two examples to validate the superconvergence results of §2.

Example 1. We consider the nonlinear Burger’s equation

$$u_{y} + uu_{x} = f, \quad (x, y) \in \Omega, \tag{3.1a}$$

where $\Omega$ is the quadrilateral $P_{1}P_{2}P_{3}P_{4}$ where $P_{1} = (0, 0)$, $P_{2} = (1, 1)$, $P_{3} = (2, 1)$ and

$$P_{4} = (2, -0.5).$$

We select the boundary conditions and $f$ such that the exact solution is

$$u(x, y) = \sqrt{3 + 2x^{2} + y^{2}}. \tag{3.1b}$$

We solve this problem on a uniform mesh having 16 elements with $p = 1, 2, 3, 4$ and plot the 0-level curves of the discontinuous Galerkin error in Figure 3.1 with Radau points marked by $x$. These results show that the solution is superconvergent at Radau points which is in full agreement with Theorem 2.3. Next we solve (3.1) with nonuniform $p$ as shown in Figure 3.2. The results shown in Figure 3.2 indicate that the superconvergence results of §2 are still valid for nonuniform polynomial degree starting with higher degree on elements at the inflow boundary of the domain and
lower polynomial degree on elements at the outflow boundary. These results are yet to be proved for nonuniform polynomial degree.

Example 2. In order to show that the superconvergence result (2.39) is optimal, we consider the linear hyperbolic problem with a reaction term

\[ u_x + 2u_y + u = f(x, y), \quad (x, y) \in [0, 1]^2, \]

(3.2a)

where the boundary conditions and \( f \) are selected such that the exact solution is

\[ u(x, y) = (1 + x + y)^7. \]

(3.2b)

We solve (3.2) on uniform meshes having 4, 16 and 36 square elements with \( p = 4 \) and plot

\[ \Psi_\Delta = \left| \int_{\Gamma_\text{out}} [1, 2] \cdot \nu (u - U) \, d\tau \right| \]

versus \( 1/h \) in Figure 3.3. As predicted by Theorem 2.4, these results show an \( O(h^{\min(2p+2, p+1)}) \) superconvergence rate of the flux on the outflow boundary of the first inflow element.

4. Conclusions. We extended the results of Adjerid and Massey [3] to nonlinear conservation laws. We proved that the discontinuous Galerkin finite element solution is \( O(h^{p+2}) \) superconvergent at the Radau points. We also showed that locally the flux is \( O(h^{2p+2}) \) superconvergent on average on the outflow boundary of the first inflow element. In the presence of reaction terms we proved similar superconvergence results for the solution at Radau points and an \( O(h^{\min(2p+2, p+1)}) \) superconvergence rate for the flux on average at the outflow boundary of the first inflow element. Furthermore, globally, we showed that, on average, the sum of the flux on the outflow boundary of the domain and the reaction term is \( O(h^{2p+1}) \) superconvergent. The flux strong superconvergence yields superconvergence of the solution at Radau points on every element. As shown in Adjerid and Massey [3], these superconvergence results for discontinuous finite element solutions may be used to construct asymptotically correct \textit{a posteriori} error estimates for steering adaptive methods. Numerical computations of Adjerid and Klauser [2] suggest that similar superconvergence results still hold for local discontinuous Galerkin solutions of convection-diffusion problems.

REFERENCES


FIG. 3.1. Zero-level curves of the error for Example 1 using $p = 1, 2, 3, 4$ (upper left to lower right). Redesh points are shown with an 'x'.


Fig. 3.2. Zero-level curves of the error for Example 1 using nonuniform polynomial degree with Radau points shown with an ‘x’ (left). Distribution of the polynomial degree of the finite element solution (right).

Fig. 3.3. The error in the flux $\Psi_\Delta$ on the outflow boundary of the first element versus $1/h$ in the log-log scale for Example 2.


