Discrete Boundary Problems via
Integro-Differential Algebra

The notion of integro-differential algebra was introduced in [Rosenkranz, M. and
Regensburger, G.: Solving and Factoring Boundary Problems for Linear Ordinary
Differential Equations in Differential Algebras, J. Symbolic Comput., 2008(43/8),
pp. 515–544] to facilitate the algebraic study of boundary problems for linear ordinary
differential equations. In this report, we construct a discrete analog in order to
investigate boundary problems for difference equations. We restrict ourselves to the
standard setting \((F, \Delta, \Sigma)\), where \(\Delta: (f_k) \mapsto (f_{k+1} - f_k)\) is the forward difference
operator and \(\Sigma: (f_k) \mapsto \left(\sum_{i=0}^{k-1} f_i\right)\) accordingly the left Riemann sum. We work here
with sequences \(f: \mathbb{Z} \to \mathbb{C}\), which we write in the variable \(k\). Key properties of the
(discrete) integro-differential algebra are proven, including the discrete analog of the
variation-of-constants formula.

Our next goal is to build up an algorithmic structure for specifying difference equations
as well as the boundary conditions, and to solve them via integro-differential
operators. We have written the relations between these operators in the form of
rewrite rules, and we prove that the resulting reduction system is Noetherian and
confluent. Thus it corresponds to a noncommutative Gröbner basis for the relation ideal of the operator ring. Let \(\mathcal{F}\) be a commutative \(K\)-algebra with \(f, g \in \mathcal{F}\) and \(\Phi\) be the set of all characters
with \(\varphi, \psi \in \Phi\). We show that every discrete operator in \(\mathcal{F}_\varphi[\Delta, \Sigma]\) can be reduced
to a linear combination of monomials \(f \varphi \Sigma g \psi \Delta^i\), where \(i \geq 0\) and each of \(f, \varphi, \Sigma, g,\)
and \(\psi\) may also be absent. Additionally, every boundary condition of \(|\Phi|\), denoting
the right ideal of \(\Phi\), has the normal form

\[
\sum_{\varphi \in \Phi} \left( \sum_{i \in \mathbb{N}} a_{\varphi, i} \varphi \Delta^i + \varphi \Sigma f_{\varphi} \right)
\]

with \(a_{\varphi, i} \in K\) and \(f_{\varphi} \in \mathcal{F}\) almost all zero. Finally, we always have the direct decom-
position \(\mathcal{F}_\varphi[\Delta, \Sigma] = \mathcal{F}[\Delta] + \mathcal{F}[\Sigma] + (\Phi)\), where \((\Phi)\) is a left \(\mathcal{F}\) module.

Using these ingredients and a given fundamental system of the difference operator,
we construct a solution algorithm for linear boundary problems over a discrete ordi-
nary integro-differential algebra (this algorithm closely resembles the corresponding
algorithm for differential equations). We conclude with an example that might be
called a discrete analog of an ill-posed boundary problem from which we extract its
Green’s function. Finally, we introduce the factorization of boundary problem with
examples.
1 Integro-Differential Algebras of Arbitrary Weight

Definition 1.1. Let \( K \) be a field and \( \lambda \in K \). We call \((\mathcal{F}, d, P)\) a differential Rota-Baxter \( K \)-algebra of weight \( \lambda \) if \( \mathcal{F} \) is a commutative \( K \)-algebra with two \( K \)-linear operators \( d : \mathcal{F} \to \mathcal{F} \) and \( P : \mathcal{F} \to \mathcal{F} \) such that the following three axioms are satisfied:

\[
d(Pf) = f \quad (1)
\]

\[
d(fg) = (df)g + f(dg) + \lambda(df)(dg) \quad (2)
\]

\[
(Pf)(Pg) = P((Pf)g) + P(f(Pg)) + \lambda P(fg) \quad (3)
\]

We refer to (1), (2), (3) as the Section Axiom, the Leibniz Axiom, and the Rota-Baxter Axiom, respectively. We call \((\mathcal{F}, d)\) a \( \lambda \)-differential algebra and \( d \) a \( \lambda \)-derivation operator; similarly, we call \((\mathcal{F}, P)\) a \( \lambda \)-Rota-Baxter algebra and \( P \) a \( \lambda \)-Rota-Baxter operator.

The Section Axiom says that \( d \circ P = 1_\mathcal{F} \), so \( P \) is required to be a section of \( d \). The differential Rota-Baxter algebra has the projectors \( J := d \circ P \) and \( E := 1_\mathcal{F} - d \circ P \), which are respectively called the initialization and the evaluation of \( \mathcal{F} \), with the vector spaces \( \mathcal{C} = \text{Ker}(d) = \text{Ker}(J) = \text{Im}(E) \) and \( \mathcal{I} = \text{Im}(P) = \text{Im}(J) = \text{Ker}(E) \), which are respectively called the constant functions and the initialized functions. Additionally, we have a canonical decomposition \( \mathcal{F} = \mathcal{C} \oplus \mathcal{I} \), which allows to split off the “constant part” of every function. Moreover, \((\mathcal{F}, P)\) is a \( \lambda \)-Rota-Baxter algebra iff \( \mathcal{I} = \text{Im}(P) \) is an ideal of \( \mathcal{F} \).

Definition 1.3. A \( \lambda \)-differential algebra \((\mathcal{F}, d)\) over a field \( K \) of characteristic zero is called ordinary if \( \dim_K \text{Ker}(d) = 1 \) and partial otherwise.

In the context of the differential Rota-Baxter algebra, \( P \) is always \( K \)-linear. On the other hand, for an integro-differential algebra, we need \( P \) to be \( \mathcal{C} \)-linear. Since the differential Rota-Baxter Algebra is ordinary when \( K \equiv \mathcal{C} \), an ordinary differential Rota-Baxter \( K \)-algebra of weight \( \lambda \) is the same as the notion of ordinary integro-differential \( K \)-algebra, \((\mathcal{F}, d, P)\).

The classical example is the algebra of continuous complex-valued functions, \( \mathcal{F} = C(\mathbb{R}) \), where for any function \( f : \mathbb{R} \to \mathbb{R} \), it follows that \( d_\lambda \) is the difference operator of weight \( \lambda \) and \( P_\lambda \) is the summation operator of weight \( \lambda \). The two operators \( d_\lambda \) and \( P_\lambda \) can be described in more details as follows.

When \( \lambda \neq 0 \), we have the discrete case with \( \lambda \)-derivation
\[ (d_\lambda f)_k = \frac{f_{k+\lambda} - f_k}{\lambda} \]

for \( \lambda > 0 \): Forward Difference,

\[ \frac{f_{k-\lambda} - f_k}{\lambda} \]

for \( \lambda < 0 \): Backward Difference,

and \( \lambda \)-Rota-Baxter operator

\[(P_\lambda f)_k = |\lambda| \sum_{s \in H(\lambda)} f_{([k]/\lambda) - s} \for \lambda > 0; \]

Left Riemann Sum,

\[ |\lambda| \sum_{s = 0}^{[k]/\lambda} |\lambda| f_{([k]/\lambda) + s} \for \lambda < 0; \]

Right Riemann Sum,

where \( s := t/\lambda \) and \( H(\lambda) \) denotes the Heaviside step function. Note that the Left/Right Riemann Sum amounts to an overestimation if \( f \) is monotonically decreasing/increasing on the interval, and an underestimation if it is monotonically increasing/decreasing.

When \( \lambda = 0 \), we may pass to the limit \( d_0 := \lim_{\lambda \to 0} d_\lambda = \frac{d}{dt} \) and \( P_0 := \lim_{\lambda \to 0} P_\lambda = f^t \), provided we restrict ourselves to functions in \( C^\infty(\mathbb{R}) \subset C(\mathbb{R}) \).

In case of integer weights \( \lambda \in \mathbb{Z} \), we can restrict ourselves to sequences \( f: \mathbb{Z} \to \mathbb{C} \). We shall write sequences in the variable \( k \). For example, \( k^2 \) denotes the sequence \([\ldots, 4, 1, 0, 1, 4, \ldots] \). Hence,

\[ (d_\lambda f)_k = \begin{cases} \frac{f_{k+\lambda} - f_k}{\lambda}, & \text{for } \lambda > 0, \\ \frac{f_{k-\lambda} - f_k}{\lambda}, & \text{for } \lambda < 0. \end{cases} \]

When \( \lambda = 1 \), we write \( \Delta := d_1 \) so that \( \Delta[f]_k = f_{k+1} - f_k \) is the standard forward difference. As with \( \lambda = -1 \), we write \( \nabla := d_{-1} \) so that \( \nabla[f]_k = f_k - f_{k-1} \) is the standard backward difference.

On the other hand,

\[ (P_\lambda f)_k = \begin{cases} \sum_{i = 1}^{[k/\lambda]} \lambda f_{[k/i] - i\lambda}, & \text{for } \lambda > 0, \\ \sum_{i = 0}^{[k/\lambda]} |\lambda| f_{[k/i] + i\lambda}, & \text{for } \lambda < 0. \end{cases} \]

When \( \lambda = 1 \), we write \( \Sigma := P_1 \) so that \( \Sigma[f]_k = \sum_{i = 0}^{k-1} f_k \) is the standard left Riemann sum. As for \( \lambda = -1 \), we write \( \Xi := P_{-1} \) so that \( \Xi[f]_k = \sum_{i = 0}^{k} f_i \) is the standard right Riemann sum.

In the sequel, we shall use the notation \( \Delta[f] \) and \( \Sigma[f] \) for delimiting the scope of the difference and summation operators. We also use the abbreviations \( f_+ := \Delta[f] + f \) to denote the forward shift and \( f_- := f - \nabla[f] \) to denote the backward shift in the case \( \lambda \neq 0 \); otherwise, \( f_+ = f_- = f \). Note that \( (f_*)_k = f_{k+1} \) and \( (f_-)_k = f_{k-1} \). Moreover, we
will use the notation $f_{r+}$ to denote the $r$-fold iteration of the *forward shift operator* and $f_{r-}$ to denote that of the *backward shift operator* on the function $f$.

## 2 Integro-Differential Algebras

Let $\mathcal{I} = \mathbb{C}^\mathbb{Z}$. From now on, we restrict ourselves to the standard settings $(\mathcal{I}, \Delta, \Sigma)$ and $(\mathcal{I}, \nabla, \Xi)$. Note that we have $u_0 = E(u)$ in this case, and in the $\lambda \neq 0$ case also $E(u_{i+}) = u_i$ and $E(u_{i-}) = u_{i-}$.

**Proposition 2.1.** The Rota-Baxter Axiom is equivalent to

$$f \Sigma[g] = \Sigma[\Delta[f] \Sigma[g]] + \Sigma[f \ast g],$$

(4)

assuming the other two axioms of Definition 1.1.

**Proof.** For proving (4), note that $\Sigma[\Delta[f] \Sigma[g]] = f \Sigma[g] - E[f \Sigma[g]] = f \Sigma[g] - 0$. Thus, $f \Sigma[g] = \Sigma[\Delta[f] \Sigma[g]]$. By the Leibniz Axiom, $f \Sigma[g] = \Sigma[\Delta[f] \Sigma[g]] = \Sigma[\Delta[f] \Sigma[g] + f \ast g] = \Sigma[\Delta[f] \Sigma[g]] + \Sigma[f \ast g]$. Conversely, assuming (4) we replace $f$ by $\Sigma[f]$ to obtain $\Sigma[f] \Sigma[g] = \Sigma[f \Sigma[g]] + \Sigma[\Sigma[f] \ast g] = \Sigma[f \Sigma[g]] + \Sigma[\Sigma[f] \Sigma[g]] + \Sigma[f \ast g]$.

For treating two-point boundary problems, it is convenient to consider two summation operators simultaneously—one initialized at the left and the other at the right boundary point. In the standard example, we have $\Sigma^*[f]_k = \sum_{i=a}^{k-1} f_i$ and $\Sigma_*[f]_k = \sum_{i=k}^{b-1} f_i$, or $\Xi^*[f]_k = \sum_{i=a}^{k} f_i$ and $\Xi_*[f]_k = \sum_{i=k}^{b} f_i$. It turns out that $(\mathcal{I}, \Delta, \Sigma^*, \Sigma_*)$ is a biintegro-differential algebra in the setting of the following definitions.

**Definition 2.2.** A biintegro-differential algebra is given by $(\mathcal{I}, \Delta, \Sigma^*, \Sigma_*)$ where both $(\mathcal{I}, \Delta, \Sigma^*)$ and $(\mathcal{I}, \Delta, -\Sigma_*)$ are ordinary integro-differential algebras.

We write $E^*f = f_a$ and $E_*f = f_b$ for their evaluations respectively $E^*$ and $E_*$. We then have

$$E^*\Sigma_* = \Sigma_* + \Sigma^* = E_*\Sigma^*,$$
where $\Sigma_* + \Sigma^*$ behaves like a definite summation since it evaluates into $K$. We further introduce the inner product $\langle | \rangle : \mathcal{F} \times \mathcal{F} \to K$ on an biintegro-differential algebra $(\mathcal{F}, \Sigma^*, \Sigma_*)$ by
\[
(f|g) = (\Sigma_* + \Sigma^*)[fg].
\]

For $\mathcal{F} = \mathcal{F}$ this gives the inner product $\langle f|g \rangle = \sum_{i=a}^{b-1} f_i g_i$.

**Proposition 2.3.** In a biintegro-differential algebra $(\mathcal{F}, \Delta, \Sigma^*, \Sigma_*)$, the operator $\Sigma^*$ is the adjoint of $\Sigma_* - 1$ with respect to $\langle | \rangle$.

**Proof.** Using the Rota-Baxter axiom (3) for $\Sigma^*$ yields
\[
\langle \Sigma^*[f]|g \rangle = E_\ast(\Sigma^*E_\ast[f]) = E_\ast(\Sigma^*[g])E_\ast([f]) - E_\ast(\Sigma^*[f\Sigma^*[g]]) - E_\ast(\Sigma^*[fg])
\]

But $\Sigma^*[g] = E_\ast(\Sigma^*[g]) - \Sigma_*[g]$ and $\langle f|E_\ast(\Sigma^*[g]) \rangle = E_\ast(\Sigma^*[g]) \langle f|1 \rangle = E_\ast(\Sigma^*[g]) E_\ast([f])$, so we rewrite the second summand as
\[
E_\ast(\Sigma^*[f\Sigma^*[g]]) = \langle f|\Sigma_*[g] \rangle = E_\ast([f])E_\ast([g]) - \langle f|\Sigma_*[g] \rangle
\]
which implies $\langle \Sigma^*[f]|g \rangle = \langle f|\Sigma_*[g] \rangle - E_\ast(\Sigma^*[f\Sigma^*[g]]) = \langle f|\Sigma_*[g] \rangle - \langle f|g \rangle = \langle f|\Sigma_*[g] \rangle - g$ as required. \(\square\)

**Lemma 2.4.** Let $(\mathcal{F}, \Delta)$ be an ordinary $\lambda$-differential algebra and $T = \Delta - c \in \mathcal{F}[\Delta]$ with an invertible solution $\tilde{u} \in \mathcal{F}$. Then $\text{Ker}(T) = [\tilde{u}]$.

**Proof.** We recall the quotient rule $\Delta[\frac{f}{g}] = \frac{g\Delta[f] - f\Delta[g]}{g^2}$.

Let $u \in \mathcal{F}$ with $Tu = 0$. Then,
\[
\Delta[\frac{u}{\tilde{u}}] = \frac{u\Delta[u] - \tilde{u}\Delta[\tilde{u}]}{u^2 + \tilde{u}^2} = \frac{cu - cu\tilde{u}}{u^2 + c\tilde{u}^2} = 0
\]

Since $\mathcal{F}$ is ordinary over a field $K$, this implies $u/\tilde{u} \in K$ so that $u \in [\tilde{u}]$. \(\square\)

**Lemma 2.5.** Let $(\mathcal{F}, \Delta)$ be a $\lambda$-differential algebra and $\tilde{u} \in \mathcal{F}$ invertible. Then
\[
T = \Delta - \Delta[\tilde{u}]/\tilde{u}
\]
is the unique monic difference operator with $[\tilde{u}] \leq \text{Ker}(T)$. If $\mathcal{F}$ is ordinary, then $[\tilde{u}] = \text{Ker}(T)$. Moreover,
\[
Tu = (\tilde{u} + \Delta[\tilde{u}])\Delta[u/\tilde{u}]
\]
for $u \in \mathcal{F}$.
Proof. It follows that $T\tilde{u} = 0$, so $[\tilde{u}] \leq \text{Ker}(T)$, and we have equality for ordinary difference algebras by Lemma 2.4. Suppose that $\tilde{T} = \Delta - c$ is any monic difference operator with $T\tilde{u} = 0$. Then, $\Delta[\tilde{u}] = c\tilde{u}$ and, hence, $c = \Delta[\tilde{u}]/\tilde{u}$. Finally, let $u \in \mathcal{F}$. Then,

$$(\tilde{u} + \Delta[\tilde{u}])\Delta[\frac{u}{\tilde{u}}] = (\tilde{u} + \Delta[\tilde{u}])\frac{\tilde{u}\Delta[u] - u\Delta[\tilde{u}]}{\tilde{u}(u + \Delta[\tilde{u}]])} = \Delta[u] - \frac{\Delta[u]}{\tilde{u}} u = Tu,$$

and the lemma is proven. \qed

More generally, we can construct for given $u(1), \ldots, u(n) \in \mathcal{F}$ with invertible Wronskian matrix, which is introduced on page 381 of [5] under the name Casorati matrix. In the discrete setting, the latter is given by

$$W[u(1), \ldots, u(n)] = \left[ \begin{array}{ccc} u(1) & \ldots & u(n) \\ u(1)_+ & \ldots & u(n)_+ \\ \vdots & & \vdots \\ u(1)_{(n-1)_+} & \ldots & u(n)_{(n-1)_+} \end{array} \right].$$

By $W[u(1), \ldots, u(n)]$, one obtains a monic differential operator such that $u(1), \ldots, u(n)$ are solutions. We write $\omega[u(1), \ldots, u(n)] = \det W[u(1), \ldots, u(n)]$ for the Wronskian determinant (also called Casorati determinant).

Whenever we can solve the homogeneous difference equation within $\mathcal{F}$, we can also solve the initial value problem for the corresponding inhomogeneous problem. The classical tool for achieving this explicitly is the discrete variation-of-constants formula (an analog of the variation-of-constants formula for differential equations), whose abstract formulation is given in Theorem 2.7 below. As usual, we will write the variation-of-constants formula for the the ring of difference operators with coefficients in $\mathcal{F}$. Let $T \in \mathcal{F}[\Delta]$ be monic (i.e. having leading coefficient 1). We call $u(1), \ldots, u(n) \in \mathcal{F}$ a fundamental system for $T$ if it is a basis for $\text{Ker}(T)$, so it yields the right number of solutions for the homogeneous differential equation $Tu = 0$. The fundamental system will be called regular if its associated Wronskian matrix is invertible in $\mathcal{F}^{n \times n}$ or equivalently if its Wronskian $\omega[u(1), \ldots, u(n)] = \det W[u(1), \ldots, u(n)]$ is invertible in $\mathcal{F}$. Of course this alone implies already that $u(1), \ldots, u(n)$ are linearly independent.

**Definition 2.6.** A monic $\lambda$-differential operator $T \in \mathcal{F}[\Delta]$ is called regular if it has a regular fundamental system.
Theorem 2.7. Let $(\mathcal{F}, \Delta, \Sigma)$ be an ordinary integro-differential algebra of weight $\lambda$. Given a regular differential operator $T \in \mathcal{F}[\Delta]$ with $\deg T = n$ and a regular fundamental system $u(1), \ldots, u(n) \in \mathcal{F}$, the canonical initial value problem

\[
Tu = f \\
u_0 = u_1 = \cdots = u_{n-1} = 0
\]

has the unique solution

\[
u = \sum_{i=1}^{n} u(i) \Sigma[\omega_+^{-1}\omega_+(i) f]
\]

so that the fundamental right-inverse is given by

\[
T^\dagger = \sum_{i=1}^{n} u(i) \Sigma \omega_+^{-1}\omega_+(i),
\]

where $\omega_+(i)$ is obtained by replacing the $i$-th column of the shifted Wronskian $\omega_+$ by the $n$-th unit vector.

This formula may also be found on page 381 of [5].

Proof. We can reformulate $Tu = f$ as a system of linear first-order difference equations with companion matrix $A \in \mathcal{F}^{n \times n}$. An $n$-th order nonhomogeneous linear difference equation

\[
u_n + a(1)\nu_{n-1} + a(2)\nu_{n-2} + \cdots + a(n)\nu = f,
\]

with real constant coefficients $a(i)$, is equivalent, via the vector $\tilde{u} = [u\ u_2\ \cdots\ u_{(n-1)}]^T$, to the linear system

\[
\tilde{u}_+ = A\tilde{u} + \tilde{f},
\]

where

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 1 \\
-a(n) & -a(n-1) & -a(n-2) & \cdots & -a(1)
\end{bmatrix}
\]

is the companion matrix and $\tilde{f} = [0, \ldots, 0, f]^T \in \mathcal{F}^n$. Setting now
\[ \ddot{u} = W \Sigma[W_+^{-1} \dot{f}], \]

we check that \( \ddot{u} \in \mathcal{F}^n \) is a solution of the first-order system \( \ddot{u}_+ = A\ddot{u} + \ddot{f} \) with initial condition \( \ddot{u}_0 = 0 \). Indeed we have \( \ddot{u}_+ = \Delta [W \Sigma[W_+^{-1} \dot{f}]] + W \Sigma[W_+^{-1} \dot{f}] = W_+ \Sigma[W_+^{-1} \dot{f}] + \ddot{f} \) by the Leibniz rule and \( AW = W_+ \) since \( u(1), \ldots, u(n) \) are solutions of \( Tu = 0 \), so the system is verified. For checking the initial condition, note that \( \mathcal{E}(\Sigma[W_+^{-1} \dot{f}]) \) is already the zero vector, so we have also \( \ddot{u}_0 = 0 \) since \( \mathcal{E} \) is multiplicative.

Writing \( W \) is moreover multiplicative. Since \( \det W \) is already the zero vector, so we have also \( \ddot{u}_0 = 0 \) since \( W \) is applicable for matrices over rings, yields \( \ddot{u}_t \) as \( \omega_+^{-1} \omega_+^t \dot{u} \) and thus

\[
\ddot{u} = (W \Sigma[\ddot{g}])_1 = \sum_{i=0}^n u(i) \Sigma[\omega_+^{-1} \omega_+^t \dot{u}] \]

since the first row of \( W \) is \([u(1), \ldots, u(n)]\). For proving uniqueness, it suffices to show that the homogeneous initial value problem only has the trivial solution. So assume \( u \) solves the given discrete boundary problem with \( f = 0 \) and choose coefficients \( c(1), \ldots, c(n) \in K \) such that

\[
u = c(1)u(1) + \cdots + c(n)u(n).\]

Then the initial conditions yield \( \mathcal{E}(Wc) = 0 \) with \( c = (c(1), \ldots, c(n))^T \in K^n \). But we have also \( \mathcal{E}(Wc) = (\mathcal{E}W)c \) because \( \mathcal{E} \) is linear, and \( \det \mathcal{E}W = \mathcal{E}(\det W) \) because it is moreover multiplicative. Since \( \det W \in \mathcal{F} \) is invertible, \( \mathcal{E}W \in K^{n \times n} \) is regular, so \( c = (\mathcal{E}W)^{-1}0 = 0 \) and \( u = 0 \).

**Example 2.8.** Solve the initial value problem of the following inhomogeneous linear difference equation

\[
y_{k+2} - 5y_{k+1} + 6y_k = 5^k - 3k, \quad y_0 = 0, \quad y_1 = 0. \tag{5}
\]

We first try to solve the homogeneous equation

\[
y_{k+2} - 5y_{k+1} + 6y_k = 0. \tag{6}
\]

The characteristic equation of (6) is \( \lambda^2 - 5\lambda + 6 = 0 \), which yields solutions \( \lambda_1 = 2 \) and \( \lambda_2 = 3 \). The general solution of the homogeneous case is thus given by \( y_k = c(1)2^k + c(2)3^k \). Since \( y_0 = 0 \) and \( y_1 = 0 \), it is trivial to see that \( c(1) = c(2) = 0 \). We now proceed to calculate the Wronskian

\[
\omega_k = \begin{vmatrix}
u_1(1) & \nu_2(1) \\
u_1(k) & \nu_2(k)
\end{vmatrix} = \begin{vmatrix}2^k & 3^k \\2^{k+1} & 3^{k+1}
\end{vmatrix} = 2^k3^{k+1} - 3^k2^{k+1} = 2^k3^k = 6^k.
\]

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It follows that \( \omega_{k+1} = 6^{k+1} \). Let us now compute the determinants \( \omega_+(i) \), which are obtained by replacing the \( i \)-th column of \( \omega_{k+1} \) by the \( n \)-th unit vector. Thus, we archive \( \omega_+(1) = \begin{vmatrix} 0 & 3^{k+1} \\ 1 & 3^{k+2} \end{vmatrix} = -3^{k+1} \) and \( \omega_+(2) = \begin{vmatrix} 2^{k+1} & 0 \\ 2^{k+2} & 1 \end{vmatrix} = 2^{k+1} \). Finally, the solution of equation (5) is given as

\[
\begin{align*}
   u_k &= \sum_{i=1}^{n} u(i)k \omega^{-1}_{k+1}(i) f_k = 2^k \sum_{i=0}^{k-1} f_i (-2^{-i-1}) + 3^k \sum_{i=0}^{k-1} f_i (3^{-i-1}) \\
   &= \sum_{i=0}^{k-1} (3^{k-i-1} - 2^{k-i-1}) f_i = \sum_{i=0}^{k-1} (3^{k-i-1} - 2^{k-i-1}) (5^i - 3i).
\end{align*}
\]

We just solved (5) using the variation-of-constant formula. To show the power of this technique, we shall solve the initial value problem once again but this time using the \( Z \)-transform. This method is described in details on page 115-127 of [4].

Take the \( Z \)-transform of (5), we achieve \( Z(y_{k+2}) - 5Z(y_{k+1}) + 6Z(y_k) = Z(5^k) - Z(3k) \), which is equivalent to \( z^2 Z(y_k) - z^2 y_0 - 3y(1) - 5z Z(y_k) + 5y_0 + 6Z(y_k) = \frac{3}{z-5} - \frac{3}{z-1}y \).

Since our initial conditions states \( y_0 = y_1 = 0 \), we can simplify the previous expression to \( (z^2 - 5z + 6) Z(y_k) = \frac{z(z^2 - 5z + 16)}{(z-5)(z-1)^2} \) or \( Z(y_k) = \frac{(z^2 - 5z + 16)}{z(z-5)(z-1)^2} \). Now, we use the method of partial fraction decomposition on the right-hand side to produce

\[
Z(y_k) = \frac{-5z}{4(z-3)} + \frac{10z}{3(z-2)} - \frac{9z}{4(z-1)} - \frac{3z}{2(z-1)^2} + \frac{z}{6(z-5)},
\]

Finally, we take the inverse \( Z \)-transform of both sides to find the solution

\[
y_k = \frac{-5}{4} 3^k + \frac{10}{3} 2^k - \frac{9}{4} - \frac{3}{2} k + \frac{1}{6} 5^k,
\]

where \( u(k) \) is the unit step function.

In this case, \( Z \)-transform demands much more computational work to find the unique solution and very sensitive to the right-hand side. Here, the partial fraction particularly requires a significant amount of multiplications and additions; moreover, this step can easily fail with more complicated right-hand side functions, e.g. \( f(y) = \ln(y^2 + 5) 5y \cos(y) - \frac{\cos(y)}{y^2 + 3y} \).

3 Integro-Differential Operators

3.1 The Algebra of Integro-Differential Operators

Definition 3.1. Let \((\mathcal{F}, \Delta, \Sigma)\) be an ordinary integro-differential algebra over a field \( K \) and \( \Phi \in \mathcal{F}^* \). The integro-differential operators \( \mathcal{F}_q[\Delta, \Sigma] \) are defined as
the \( K \)-algebra generated by the symbols \( \Delta \) and \( \Sigma \), the "functions" \( f \in \mathcal{F} \# \) and the "characters" \( \varphi \in \Phi \cup \{E\} \), modulo the rewrite rules given in Table 3.1. If \( \Phi = \mathcal{F}^* \), we write simply \( \mathcal{F}_\Phi[\Delta, \Sigma] \).

\[
\begin{array}{c|c|c}
 f \cdot g \rightarrow f \cdot g & \Delta f \rightarrow \Delta[f] + f_+ \Delta & \Delta \varphi \rightarrow 0 \\
\varphi \psi \rightarrow \psi & \Sigma f \Sigma \rightarrow \Sigma[f] \Sigma - \Sigma \Sigma[f] - \Sigma f & \Sigma \Delta f \rightarrow f_- - \Sigma \Delta[f_-] - E(f_-) E \\
\varphi \rightarrow \varphi(f) \varphi & \Delta \Sigma \rightarrow 1 & \Sigma f \varphi \rightarrow \Sigma[f] \varphi
\end{array}
\]

Table 3.1. Rules for the Discrete Integro-Differential Algebra

### 3.2 Parametrized Noncommutative Gröbner Bases

Let \( X := \{\Delta, \Sigma, E\} \cup \mathcal{F}^* \cup \Phi \) where \( \mathcal{F}^* \cup \{1\} \) is a \( K \)-basis of \( \mathcal{F} \) and \( \Phi \) is the set of all characters. Then, we use the following theorem.

**Theorem 3.2.** Let \( Q \) be a reduction system for \( K(X) \) and let \( \leq \) a Noetherian monoid partial order on \( X \), compatible with \( Q \). Then the following conditions are equivalent:
1. All ambiguities of \( Q \) are resolvable.
2. All ambiguities of \( Q \) are resolvable relative to \( \leq \).
3. All elements of \( K(X) \) are reduction-unique under \( Q \).
4. We have the direct decomposition \( K(X) = K(X)_{\text{irr}} + I_Q \) as \( K \)-modules.

**Proof.** This theorem is called the *Diamond Lemma for Ring Theory* in Bergman’s homonymous paper [1]. Hence, see the proof for Theorem 1.2 in [1]. \( \square \)

**Proposition 3.3.** For every ordinary integro-differential algebra \((\mathcal{F}, \Delta, \Sigma)\) over a field \( K \) and for all \( \Phi \subseteq \mathcal{F}^* \), the rules of Table 3.1 constitute a convergent reduction system on the corresponding free \( K \)-algebra.

**Proof.** We write \( Q \) for the reduction system described by Table 3.1. Using Theorem 3.2, we construct a Noetherian monoid partial order on \( X \) that is compatible with \( Q \), and prove that all ambiguities of \( Q \) are resolvable. We put \( f < \Delta \) for all \( f \in \mathcal{F}^* \), extended to words by the graded lexicographic construction. The resulting partial order is Noetherian (since it is on the generators) and compatible with the monoid structure (by its grading). It respects the reduction system \( Q \) because all rules reduce the word length except for the Leibniz rule, which is compatible because of \( f < \Delta \). Onto the ambiguities of \( Q \) are resolvable, we must consider 14 overlap ambiguities. In our case, there is no inclusion ambiguities. The calculation is easy in all 14 cases,
using also the axioms of integro-differential algebras for $\mathcal{F}$ and table 5.1. As a representative example, let us compute the $\mathcal{I}$-polynomial of the two reduction rules $\sigma = (\Sigma f \Sigma, \Sigma [f] \Sigma - \Sigma \Sigma [f] - \Sigma f)$ and $\tau = (\Sigma g \Delta, g_\Sigma - \Sigma \Delta [g] - E(g_\Sigma) E)$ as

$$\Sigma [f] \Sigma g \Delta - \Sigma [f] g \Delta - \Sigma f g \Delta - \Sigma f g_\Sigma + \Sigma f \Sigma \Delta [g]_\Sigma + \Sigma f E(g_\Sigma) E$$

$$= \Sigma [f] g_\Sigma - \Sigma [f] \Sigma g - \Sigma [f] \Sigma \Sigma [f] g \Delta - \Sigma f g \Delta - \Sigma f g_\Sigma + \Sigma f \Sigma \Delta [g]_\Sigma + \Sigma [f] E(g_\Sigma) E$$

$$= \Sigma [f] g_\Sigma - \Sigma [f] \Sigma g + \Sigma [f] \Sigma g_\Sigma - (\Sigma [f] g)_\Sigma + \Sigma \Delta [\Sigma [f] g]_\Sigma + E((\Sigma [f] g)_\Sigma) E - f_\Sigma - \Sigma [f] \Delta [g]_\Sigma - \Sigma f \Delta [g]_\Sigma$$

meaning the overlap ambiguity $(\sigma, \tau, \Sigma f, \Sigma g \Delta)$ is resolvable.

As another representative example, let us compute the $\mathcal{I}$-polynomial of the two reduction rules $\sigma = (\Sigma f \Sigma, \Sigma [f] \Sigma - \Sigma \Sigma [f] - \Sigma f)$ and $\tau = (\Sigma f \varphi, \Sigma [f] \varphi)$ as

$$\Sigma [f] \Sigma g \varphi - \Sigma [f] g \varphi - \Sigma f g \varphi - \Sigma f \Sigma [g] \varphi$$

$$= \Sigma [f] \Sigma g \varphi - \Sigma [\Sigma [f] g] \varphi - \Sigma [f g] \varphi - \Sigma [f \Sigma [g]] \varphi$$

$$= \Sigma [f] \Sigma g \varphi - (\Sigma [\Sigma [f] g] + [f g] + \Sigma [f \Sigma [g]]) \varphi$$

$$= 0,$n

meaning the overlap ambiguity $(\sigma, \tau, \Sigma f, \Sigma g \varphi)$ is resolvable.

\[\square\]

### 3.3 Normal Forms for Integro-Differential Operators

**Lemma 3.4.** Every integro-differential operator in $\mathcal{F}_\Phi [\Delta, \Sigma]$ can be reduced to a linear combination of monomials $f \varphi \Sigma g \psi \Delta^i$, where $i \geq 0$ and each of $f, \varphi, \Sigma, g$, and $\psi$ may also be absent.

**Proof.** We call a monomial consisting only of functions $f$’s and functionals $\varphi$’s al-
gebraic. Using the left column of Table 3.1, we can immediately show that such monomials can be reduced to \( f \) or \( \varphi \) or \( f\varphi \).

Hence, let \( w \) be an arbitrary monomial in \( \mathcal{F}^\# \cup \Phi \cup \{ \Delta, \Sigma \} \). By using the middle column of Table 3.1, we may move all \( \Delta \)'s right, so that all monomials \( w \) have the form \( w_1 w_2 \ldots w_n \Delta^i \) with \( i \geq 0 \) and \( w_1, w_2, \ldots, w_n \in \mathcal{F}^\# \cup \Phi \cup \{ \Delta, \Sigma \} \).

We may further assume that there is at most one occurrence of \( \Sigma \) among the \( w_1, w_2, \ldots, w_n \). Otherwise, the monomials \( w_1, w_2, \ldots, w_n \) contain \( \Sigma \tilde{w} \Sigma \), where each \( \tilde{w} = f\varphi \) is an algebraic monomial, which then can be reduce

\[
\Sigma \tilde{w} \Sigma \rightarrow (\Sigma f \varphi) \Sigma \rightarrow \Sigma [f] \varphi \Sigma
\]

by using the corresponding rule of Table 3.1. Applying these rules repeatedly, we arrive at algebraic monomials left and right of a single \( \Sigma \). In the case where \( \Sigma \) is absent, we will only achieve a single algebraic monomial. \( \square \)

**Definition 3.5.** The elements of the right ideal

\[
|\Phi| = \Phi \cdot \mathcal{F}_\Phi[\Delta, \Sigma]
\]

are called **Stieltjes boundary conditions** over \( \mathcal{F} \); if there is no danger of ambiguity, we will henceforth just speak of “boundary conditions”.

**Proposition 3.6.** Every boundary condition of \( |\Phi| \) has the normal form

\[
\sum_{\varphi \in \Phi} \left( \sum_{i \in \mathbb{N}} a_{\varphi,i} \varphi \Delta^i + \varphi \Sigma f_{\varphi} \right)
\]

with \( a_{\varphi,i} \in K \) and \( f_{\varphi} \in \mathcal{F} \) almost all zero.

**Proof.** By Lemma 3.4, every boundary condition of \( |\mathcal{F}| \) is a linear combination of monomials of the form

\[
w = \chi f \varphi \Sigma g \psi \Delta^i \quad \text{or} \quad w = \chi f \varphi \Delta^i
\]

where each of \( f, g, \varphi, \psi \) may also be missing.

Using the left column of Table 3.1, the prefix \( \chi f \varphi \) can be reduced to a scalar multiple of a functional: \( \chi f \varphi \rightarrow \chi(f) \chi \varphi \) or \( \chi f \rightarrow \chi(f) \chi \). Hence, without losing generosity, we may assume that \( f \) and \( \varphi \) are not present. This finishes the right-hand case of (8), since \( w = \chi \Delta^i \) is already in normal form.
For the remaining case $w = \chi \Sigma g \psi \Delta^i$, we have two cases. In the first case, $\psi$ is present. Then, it follows
\[
\chi \Sigma g \psi \to \chi [g] \psi \to \chi (\Sigma [g]) \chi \psi \to \chi \Sigma [g] \psi \in K,
\]
so $w$ is a scalar multiple of $\psi \Delta^i$, and we are done. Finally, for the second case where $\psi$ is absent, we have $w = \chi \Sigma g \Delta^i$. If $i = 0$, this is already a normal form. Hence, we assume $n > 0$; it follows that
\[
w = \chi \Sigma g \Delta \Delta^{i-1} = \chi (\Sigma g \Delta) \Delta^{i-1} = \chi (g_\cdot) \chi \Delta^{i-1} - \chi \Sigma \Delta[g_\cdot] \Delta^{i-1} - E(g_\cdot) E \Delta^{i-1}
\]
where the first and the last summand are in the required normal form. The middle summand is to be reduced recursively, eventually leading to a middle term in normal form. The right-hand case yields $w \in \mathcal{F}[\Delta]$. In the left-hand side case, if $\psi$ is present, we may reduce $\Sigma g \psi$ to $\Sigma [g] \psi$, which again shows that $w \in (\Phi)$. Hence, we are left with $w = f \Sigma g \Delta^i$, and we may assume $i > 0$ since otherwise we have $w \in \mathcal{F}[\Sigma]$ immediately. Otherwise, we can reduce
\[
w = f \Sigma g \Delta \Delta^{i-1} = f (\Sigma g \Delta) \Delta^{i-1}
\]
\[
= f (g_\cdot - \Sigma \Delta[g_\cdot] - E(g_\cdot) E) \Delta^{i-1}
\]
\[
= (f \cdot g_\cdot) \Delta^{i-1} - f \Sigma \Delta[g_\cdot] \Delta^{i-1} - E(g_\cdot) f E \Delta^{i-1},
\]
where the first summand is in $\mathcal{F}[\Delta]$ and the last summand in $(\Phi)$. The middle term may be reduced recursively until the exponent of $\Delta$ has become zero, leaving a term in $\mathcal{F}[\Sigma]$. \qed

**Proposition 3.7.** For an ordinary integro-differential algebra $\mathcal{F}$ and characters $\Phi \subseteq \mathcal{F}^\bullet$, we have the direct decomposition $\mathcal{F}[\Delta, \Sigma] = \mathcal{F}[\Delta] + \mathcal{F}[\Sigma] + (\Phi)$, where $(\Phi)$ is a left $\mathcal{F}$ module.

**Proof.** Using Table 3.1, we can trivially confirm that all integro-differential operators having the described sum representation $T + G + P$ are indeed in normal form. Let us now prove that every monomial $w \in \mathcal{F}[\Delta, \Sigma]$ can be written as $T + G + P$ in a unique way. If $w$ starts with a character, we obtain a boundary condition, such that $w \in (\Phi) \subseteq (\Phi)$, and we are done. Hence, we assume this is not the case. According to Lemma 3.4 we know that $w = f \varphi \Sigma g \psi \Delta^i$ or $w = f \varphi \Delta^i$,
\[
\text{where each of } g, \varphi, \psi \text{ may also be absent. Yet, } w \in (\Phi) \text{ unless } \varphi \text{ is missing}. \text{ Hence, we can further assume}
\]
\[
w = f \Sigma g \psi \Delta^i \text{ or } w = f \Delta^i.
\]
The right-hand case yields $w \in \mathcal{F}[\Delta]$. In the left-hand side case, if $\psi$ is present, we may reduce $\Sigma g \psi$ to $\Sigma [g] \psi$, which again shows that $w \in (\Phi)$. Hence, we are left with $w = f \Sigma g \Delta^i$, and we may assume $i > 0$ since otherwise we have $w \in \mathcal{F}[\Sigma]$ immediately. Otherwise, we can reduce
\[
w = f \Sigma g \Delta \Delta^{i-1} = f (\Sigma g \Delta) \Delta^{i-1}
\]
\[
= f (g_\cdot - \Sigma \Delta[g_\cdot] - E(g_\cdot) E) \Delta^{i-1}
\]
\[
= (f \cdot g_\cdot) \Delta^{i-1} - f \Sigma \Delta[g_\cdot] \Delta^{i-1} - E(g_\cdot) f E \Delta^{i-1},
\]
where the first summand is in $\mathcal{F}[\Delta]$ and the last summand in $(\Phi)$. The middle term may be reduced recursively until the exponent of $\Delta$ has become zero, leaving a term in $\mathcal{F}[\Sigma]$. \qed
4 Solving Ordinary Boundary Problems

4.1 The Solution Algorithm

For convenience, we will use the notation $S = \Delta + 1$, $A = \Sigma^* = (E^* - 1)\Sigma_*$, $B = \Sigma_* = (E_+ - 1)\Sigma^*$, $L = E^*$, and $R = E_*$

A discrete boundary problem is specified by a difference operator $T$ and a boundary space $B = [\beta(1), \ldots, \beta(n)]$ generated by $n$ Stieltjes conditions $\beta(1), \ldots, \beta(n) \in |F^*|$. In traditional notation, the boundary problem $(T, B)$ is then given by

$$
\begin{align*}
Tu &= f \\
\beta(1)(u) &= \cdots = \beta(n)(u) = 0
\end{align*}
$$

We can now give a solution algorithm for computing $G = (T, B)^{-1}$, provided we have a regular fundamental system $u(1), \ldots, u(n)$ for $Tu = 0$ and a $K$-basis $\beta(1), \ldots, \beta(n)$ for $B$. The algorithm proceeds in three steps:

1. Construct the fundamental right inverse $T^* = \sum_{i=1}^{n} u(i) \sum_{k=1}^{i} \omega_{k,i}^{-1} \omega(i)_{k+1}$

2. Determine the projector $P = \sum_{i=1}^{n} u(i) \tilde{\beta}(i) \in \mathcal{F}[\Delta, \Sigma]$ as in Theorem 4.17. on page 34 of [6].

3. Compute $G = (1 - P)T^* \in \mathcal{F}[\Delta, \Sigma]$

Example 4.2: Extract the Green’s operator from equation (6). From previous calculation, we acquire the fundamental right-inverse

$$(T^* f)_k = \sum_{i=0}^{k-1} (3^{k-i-1} - 2^{k-i-1}) f_i.
$$

Let us proceed to computing the kernel projector $P$. With solutions $u(1)_k = 2^k$ and $u(2)_k = 3^k$ and boundary conditions $\beta(1)(u_k) = u_0 = 0$ and $\beta(2)(u_k) = u_1 = 0$, we construct the evaluation matrix $V = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, which has the respective inverse matrix $V^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$. Hence, $\tilde{\beta}(1) = 3\beta(1) - \beta(2)$ and $\tilde{\beta}(2) = -2\beta(1) + \beta(2)$. As a result, we achieve the action of the Green’s operator

$$(Gf)_k = ((1 - P)T^* f)_k = (T^* f)_k - \tilde{\beta}(1)(T^* f)_k u(1)_k - \tilde{\beta}(2)(T^* f)_k u(2)_k$$

$$= (T^* f)_k \sum_{i=0}^{k-1} (3^{k-i-1} - 2^{k-i-1}) f_i.$$

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Finally, we once again can extract the Green’s operator as
\[ G = 3^{k-1} \sum 3^{-k} - 2^{k-1} \sum 2^{-k}. \]

**Example 4.3.** Consider the following example where we extract the Green’s Operator from an ill-posed discrete boundary problem.

\[
\begin{array}{c|c}
\ \ u_{n+1} - 2u_n &= f_n \\
\ \ u_2 &= 0
\end{array}
\]

We first solve for the missing boundary conditions \( u_0 \) and \( u_1 \). Since \( u_2 = 0 \), it follows that \( u_1 = -\frac{l_0}{2} \), which leads to \( u_0 = -\frac{l_0}{2} - \frac{l_1}{4} \). Then, we have

\[
u_n = f_{n-1} + 2u_{n-1} = f_{n-1} + 2(f_{n-2} + 2u_{n-2})
= f_{n-1} + 2f_{n-2} + 4u_{n-2} = f_{n-1} + 2f_{n-2} + 4f_{n-3} + 8u_{n-3}
\vdots
= \sum_{k=1}^{n-1} 2^{k-1} f_{n-k} + 2^{n-1} \left( -\frac{l_1}{2} \right) ^{n-1} \sum_{k=1}^{n} 2^{k-1} f_{n-k} - 2^{n-1} f_0 - 2^{n-2} f_1.
\]

At this step, we perform an index reflection with \( l = n - k \) and \( k - 1 = n - l - 1 \), then replace \( l \) with \( k \). As a result, we have \( u_n = 2^{n-1} \sum_{k=0}^{n-1} 2^{-k} f_k - 2^{n-1} f_0 - 2^{n-2} f_1 \). Thus, the Green’s Operator in this case is

\[ G = 2^{k-1} \sum 2^{-k} - 2^{k-1} E - 2^{k-2} E(\Delta - 1). \]

**Corollary 4.4.** Let \((\mathcal{F}, \Delta, \Sigma)\) be an ordinary integro-differential algebra and consider the \(\lambda\)-differential operator \( T = (S - \lambda_1) \ldots (S - \lambda_n) \in \mathcal{F}[S] \subset \mathcal{F}[\Delta] \) with \( \lambda_1, \ldots, \lambda_n \in K \) are mutually distinct. Assume each \( S[u] = \lambda_i u, E(u) = 1 \) has a solution \( u = \lambda_i^{-k} \in \mathcal{F} \) with reciprocal \( u^{-1} = \lambda_i^{-k} \in \mathcal{F} \). Then, we have

\[ T^\bullet = \sum_{i=1}^{n} \mu_i \lambda_i^k \Sigma \lambda^{-k}, \]

where \( \mu_i = (\lambda_i - \lambda_1) \ldots (\lambda_{i-1} - \lambda_1)(\lambda_{i+1} - \lambda_1) \ldots (\lambda_i - \lambda_n). \)

**Proof.** Let us write \( V \) for the \( n \times n \) Vandermonde determinant in \( \lambda_1, \ldots, \lambda_n \) and \( V_i \) for the \((n-1)\times(n-1)\) Vandermonde determinant in \( \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n \). Evaluating the quantities of \( T^\bullet = \sum_{i=1}^{n} u(i) \Sigma \omega_+^{-1} \omega_+(i) \), one sees immediately that

\[ \omega_+ = \lambda_1^k \ldots \lambda_n^k V_+ \text{ and } \omega(i)_+ = (-1)^{n+i} \lambda_1^k \ldots \lambda_{i-1}^k \lambda_{i+1}^k \ldots \lambda_n^k V(i)_+. \]
Hence we have $\omega^{-1}\omega(i)_+ = (-1)^{n+i}\lambda^{-k}V(i)_+ = V_+$. Using the well-known formula for the Vandermonde determinant, one obtains $\omega^{-1}\omega(i)_+ = \mu_i \lambda^{-k}$, and now the result follows from Theorem 2.7.

**Proposition 4.5.** Let $(S-h,[\beta])$ be a boundary problem over the integro-differential algebra $\mathcal{F}[a,b]$. This boundary problem is regular iff $c = \beta \cdot h^k$ is nonzero. In this case, the Green’s operator is $(1-c^{-1}h^k\beta)h^{k-1}\Sigma h^{-k}$.

**Proof.** The regularity criterion $c = \beta(h^k) \neq 0$ follows directly from Proposition 4.16 of [6]. Now we proceed according to the solution algorithm given above. One may use Theorem 2.7 or direct verification for seeing that $(S-h)^{\downarrow} = h^{k-1}\Sigma h^{-k}$. Since $\tilde{\beta} = c^{-1}\beta$, we obtain $P = c^{-1}h^k\beta$, and the formula for the Green’s operator follows.

Applying Proposition 4.5 to the ill-posed boundary problem in Example 4.3, one may compute its Green’s operator.

$$G = (1 - c^{-1}h^k\beta)h^{k-1}\Sigma h^{-k}$$

$$= (1 - h^{-2}h^k E S^2)h^{k-1}\Sigma h^{-k}$$

$$= h^{k-1}\Sigma h^{-k} - h^{k-3} E S^2 h^k \Sigma h^{-k}$$

$$= h^{k-1}\Sigma h^{-k} - h^{k-3} E h^{k+2} (\Sigma + 1 + S) h^{-k}$$

$$= h^{k-1}\Sigma h^{-k} - h^{k-3} E h^{k+2} h^{-k} - h^{k-3} E h^{k+2} S h^{-k}$$

$$= h^{k-1}\Sigma h^{-k} - h^{k-1} E - h^{k-2} E S.$$

Here, our rewrite rules and normal form allow us to simplify the Green operator and terminate the reduction. This answer also appeared to match our answer using algebraic method in Example 4.3., thus confirming our solution algorithm in this particular case.

### 4.2 Factorization of Boundary Problems

**Lemma 4.6.** Let $(T, \mathcal{B})$ be a regular boundary problem. For every factorization $T = T_1 T_2$ into regular differential operators there is a boundary space $\mathcal{B}_2 \leq \mathcal{B}$ that makes $(T_2, \mathcal{B}_2)$ a regular boundary problem.

**Proof.** Proof replicated from Lemma 6.12 of [6].

**Theorem 4.7.** Given $(T, \mathcal{B}) \in \mathcal{F}[^\Delta]_\Phi$, every factorization $T = T_1 T_2$ into regular discrete operators can be lifted to a factorization $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$ of the given boundary problem such that $(T_1, \mathcal{B}_1), (T_2, \mathcal{B}_2) \in \mathcal{F}[^\Delta]_\Phi$ and $\mathcal{B}_2 \leq \mathcal{B}$. 

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Proof. Proof replicated from Theorem 6.15. of [6].

Example 4.8. In computing \( \mathcal{B}_2 = [\omega(1), \ldots, \omega(n)] \) we have used particular (ordered) bases for \( \ker(T) \) and for \( \mathcal{B} \). Different bases lead to different boundary spaces \( \mathcal{B}_2 \) for \( T_2 \), as in the following simple example \((S^2 - 5S + 6, [L, R])\) in \( \mathcal{F}[S, A, B] \) with \( \mathcal{F}[0, 1] \). Using \( v(1)_k = 2^k \), \( w(1)_k = 3^k \) as a basis of \( \ker(S^2 - 5S + 6) \) and the given boundary conditions \( \beta(1) = L, \beta(2) = R \) as a basis for \( \mathcal{B} \), we have \( \beta(v, w) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \) so that

\[
C \cdot \beta(w) = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \omega \\ \varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} L \\ R \end{pmatrix} = \begin{pmatrix} L \\ R - 3L \end{pmatrix}.
\]

Thus, we obtain \((T_2, \mathcal{B}_2) = (S - 3, [L])\) in this case. However, if we reorder the basis of \( \mathcal{B} \) as \( \beta_1 = R, \beta_2 = L \), we would end up with \( \omega = R \) and \( \varphi = L - 3R \) so that we have now \((T_2, \mathcal{B}_2) = (S - 3, [R])\).

Based on the results in Proposition 4.5, their Green’s operators are respectively \( G_2 = 3^{k-1}(1 - L)\Sigma 3^{-k} \) and \( G_2 = 3^{k-1}(1 - R)\Sigma 3^{-k} \), hence we get in both cases \( \mathcal{B}_1 = [(R - 3L)3^{k-1}(1 - L)\Sigma 3^{-k}] = [(L - 3R)3^{k-1}(1 - R)\Sigma 3^{-k}] = [E] \) for the left factor. Altogether we have thus achieved the factorizations

\[
(S^2 - 5S + 6, [L, R]) = (S - 2, [E]) \cdot (S - 3, [L]) = (S - 2, [E]) \cdot (S - 3, [R])
\]

or

\[
\begin{array}{ccc}
\begin{array}{ccc}
& u_{k+2} - 5u_{k+1} + 6u_k &= f_k \\
& u_0 = u_1 &= 0
\end{array} & \cdot & \begin{array}{ccc}
& u_{k+1} - 2u_k &= f_k \\
& u_0 &= 0
\end{array} & \cdot & \begin{array}{ccc}
& u_{k+1} - 3u_k &= f_k \\
& u_0 &= 0
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{ccc}
& u_{k+2} - 5u_{k+1} + 6u_k &= f_k \\
& u_0 = u_1 &= 0
\end{array}
\end{array}
\]

in more intuitive notation.

Example 4.9. Again we work in \( \mathcal{F}[S, A, B] \) with \( \mathcal{F}[0, 1] \). We consider the boundary problem \( \mathcal{P} = (S^4 + 4, [L, R, RS, RS^2]) \), written in the traditional formulation as

\[
\begin{array}{ccc}
& u_{k+4} + 4u_k &= f_k \\
& u_0 = u_1 = u_2 = u_3 &= 0
\end{array}
\]

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We employ the natural factorization $S^4 + 4 = (S^2 - 2i)(S^2 + 2i) = (S - 1 - i)(S + 1 + i)(S - 1 + i)(S + 1 - i)$. Using $a_k = (1 + i)^k$, $b_k = (-1 - i)^k$, $c_k = (-1 + i)^k$, $d_k = (1 - i)^k$ as a basis of $\ker(S^4 - 4S)$ and the given boundary conditions $\beta(1) = L$, $\beta(2) = R$, $\beta(3) = RS$, $\beta(4) = RS^2$ as a basis for $B$, we have

$$\beta(a,b,c,d) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 + i & -1 - i & -1 + i & 1 - i \\
2i & 2i & -2i & -2i \\
-2 + 2i & 2 - 2i & 2 + 2i & -2 - 2i
\end{pmatrix}$$

so that

$$C \cdot \beta(c,d) = \begin{pmatrix}
0 & 0 & \frac{3}{4}i & -\frac{1}{8} + \frac{3}{8}i \\
0 & 0 & \frac{3}{4}i & \frac{1}{8} - \frac{3}{8}i \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1 & 1 & 1 \ \\
-1 + i & 1 - i \ \\
-2i & -2i \ \\
2 + 2i & -2 - 2i
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}$$

and

$$\begin{pmatrix}
0 & 0 & \frac{3}{4}i & -\frac{1}{8} + \frac{3}{8}i \\
0 & 0 & \frac{3}{4}i & \frac{1}{8} - \frac{3}{8}i \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
L \\
R \\
RS \\
RS^2
\end{pmatrix} = \begin{pmatrix}
\frac{1}{4}iRS + (-\frac{1}{8} + \frac{3}{8}i)RS^2 \\
\frac{1}{4}iRS + (\frac{1}{8} - \frac{3}{8}i)RS^2 \\
\frac{1}{4}iRS + (-\frac{1}{8} + \frac{3}{8}i)RS^2 \\
\frac{1}{4}iRS + (\frac{1}{8} - \frac{3}{8}i)RS^2
\end{pmatrix}.$$

