Study of Spectral Problems Arising in the Theory of Optical Fibers

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1 Introduction

In this paper we investigate the Zakharov-Shabat system of differential equations underlying spectral theory in optical fibers. The eigenvalues presented in the Zakharov-Shabat system can have imaginary or real values depending on the structure of the equation and how the pulses are broken up graphically. This is the Zakharov-Shabat system of differential equations:

\[ u'_1(t) = -i\xi v_1 + q(t)v_2 \]
\[ u'_2(t) = -q(t)v_1 + i\xi v_2 \]

where \( q(t) \) is a real potential and \( \xi \) is a constant, complex number and represents the spectral parameter. We also have

\[ \xi = \frac{v}{2} + \frac{A}{2} i. \]

This spectral problem has relation with the nonlinear Schrödinger (NLS) equation which constitutes pulses in the optical fibers and gives information on localized initial conditions. One form of the NLS equation is

\[ iu_t = -\frac{1}{2} u_{ttt} - |u|^2 u. \]

Optical solitons are solutions that are theoretical and are focused on the NLS equation. These solutions are either periodic in the variable or maintain their exact shape and don't lose fibers, which is optimal. These soliton solutions correspond to the Zakharov-Shabat system of equations (Shaw, 2004). A typical soliton for this system is

\[ u(\zeta, \tau) = A\text{sech}[A(\tau - v\zeta)] e^{i\left[v\tau - (v^2 - A^2)\zeta^2 \right]/2}. \]

A represents the amplitude as well as the pulse width of the soliton and \( v \) represents its speed as well as the frequency shift. There is a connection, here, between \( u \) and \( q \) as \( u(0, t) = q(t) \). Thus, \( q(t) \) can allow us find the eigenvalues. In inverse scattering theory, there are ways to find \( q(t) \) just from knowing the eigenvalues and the spectrum in question. When we set \( \zeta = 0 \) and choose \( v = 0 \), we get a purely imaginary eigenvalue. This creates a unique result that is graphically
different from the other cases. There are several different types of solutions and eigenvalues that can be examined in a system of this form, and we will be examining several of these.

2 Background

When examining these equations, we want to keep the ones that decay rather than the ones that grow and diverge. Those that decay give us information on the eigenvalues of the system and where the system has optimal parameters. Those that grow exponentially, however, do not give valuable information about the system.

In the case we are looking at, our system

\[ v'(t) = \begin{bmatrix} -i\xi & h \\ -h & i\xi \end{bmatrix} v(t) \]

has the form \( \begin{bmatrix} \alpha & \beta \\ -\beta & -\alpha \end{bmatrix} \). The eigenvalues for this are

\[ \lambda_{1,2} = \pm \sqrt{\alpha^2 - \beta^2} = \pm \sqrt{-\xi^2 - h^2} = \pm i\sqrt{\xi^2 + h^2}. \]

The eigenvectors for each eigenvalue are,

\[ \lambda_1 = i\sqrt{\xi^2 + h^2}; \quad z_1 = \left( \frac{-i\sqrt{\xi^2 + h^2} + i\xi}{h} \right). \]

\[ \lambda_2 = -i\sqrt{\xi^2 + h^2}; \quad z_2 = \left( \frac{i\sqrt{\xi^2 + h^2} + i\xi}{h} \right). \]

Diagonalizing \( \begin{bmatrix} \alpha & \beta \\ -\beta & -\alpha \end{bmatrix} \), we let \( \gamma = \sqrt{\xi^2 + h^2} \), thus

\[ e^{t\begin{bmatrix} \alpha & \beta \\ -\beta & -\alpha \end{bmatrix}} = \begin{bmatrix} \cos \gamma t - \frac{i\xi}{\gamma} \sin \gamma t & \frac{h}{\gamma} \sin \gamma t \\ -\frac{h}{\gamma} \sin \gamma t & \cos \gamma t + \frac{i\xi}{\gamma} \sin \gamma t \end{bmatrix} = \phi(t). \tag{1} \]

We will use this result as we examine the decaying and diverging solutions.

Looking at the system graphically, we can tell what type of eigenvalues it has. If there is one smooth bump in the graph, this means that there will be purely imaginary eigenvalues. These are called single lobe potentials, meaning that \( q(t) \geq 0 \) increases on the left hand side of \( t = 0 \) and decreases on the right hand side. We assume that \( t = 0 \) is the center of concentration of energy in the lobe.

On the other hand, when there are multiple bumps shown the potentials may be complex-valued and may have nonimaginary eigenvalues. Depending on the shape of the corresponding graph of \( q(t) \), the eigenvalues either move down the imaginary axis and into the lower half of the \( \xi \)-plane, or there are pairs of eigenvalues that collide with each other and shift each other off the imaginary axis (Hasegawa, 1990).
3 Examples

First, here is a scenario where \( q(t) = 0 \) when \( t \leq -d \), \( q(t) = h \) when \( -d \leq t \leq d \), and \( q(t) = 0 \) when \( d \leq t \).

![Diagram](image)

We will split this scenario into three sections based off the piecewise nature of this equation.

a) In the first section where \( q(t) = 0 \) when \( t \leq -d \), we get that \( v'_1(t) = -i\xi v_1 \) and \( v'_2(t) = i\xi v_2 \). This means that the vector solution is \( v_1(t) = c_1 e^{-i\xi t} \) and \( v_2(t) = c_2 e^{i\xi t} \).

Now we will assume that \( Im(\xi) > 0 \). This shows us the following results:

\[
\begin{align*}
v_1(t) &= c_1 e^{-i\xi t} \rightarrow 0 \text{ as } t \rightarrow -\infty \\
v_2(t) &= c_2 e^{i\xi t} \rightarrow \infty \text{ as } t \rightarrow -\infty.
\end{align*}
\]

Thus we keep \( v_1(t) \) since it decays and converges to 0, and we reject \( v_2(t) \) because it grows and diverges.

b) In the second region where \( q(t) = h \) when \( -d \leq t \leq d \), we get that \( v'_1(t) = -i\xi v_1 + hv_2 \) and \( v'_2(t) = -hv_1 + i\xi v_2 \). We can convert this into a matrix system of the form

\[
\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = \begin{bmatrix} -i\xi & h \\ -h & i\xi \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\]

From this, we will use \( \phi(t) \) and \( e^{t \begin{bmatrix} \alpha & \beta \\ -\beta & -\alpha \end{bmatrix}} \) from (1) and let \( \alpha = -i\xi \) and \( \beta = h \) to get the following results.

\[
v(t) = \phi(t + d)v(-d)
\]
c) Lastly, the third region where \( q(t) = 0 \) when \( d \leq t \), gives us, similarly to the first region, that \( v'_1(t) = -i\xi v_1 \) and \( v'_2(t) = i\xi v_2 \). Thus, the vector solution is also \( v_1(t) = c_1 e^{-i\xi t} \) and \( v_2(t) = c_2 e^{i\xi t} \). This shows us the following results:

\[
\begin{align*}
v_1(t) &= c_1 e^{-i\xi t} \to \infty \text{ as } t \to \infty \\
v_2(t) &= c_2 e^{i\xi t} \to 0 \text{ as } t \to \infty, \text{ where } \text{Im}(\xi) > 0.
\end{align*}
\]

Thus we reject \( v_1(t) \) since it grows and diverges to \( \infty \), and we keep \( v_2(t) \) because it decays and converges.

Now, looking more closely at this problem, let’s let \( \xi = \alpha + i\beta, \beta > 0 \). So,

\[
\begin{align*}
v_1(t) &= c_1 e^{-i(\alpha + i\beta)t} = c_1 e^{-i\alpha t} e^{i\beta t} \\
v_2(t) &= c_2 e^{i(\alpha t + i\beta)t} = c_2 e^{i\alpha t} e^{-\beta t}.
\end{align*}
\]

For this, we see that as \( t \to \infty \), \( v_1(t) \) grows and \( v_2(t) \) decays.

Deriving these functions for the eigenvalue \( \xi \), we find there is decay at both ends of the graph. The eigenvalues tell us where the system of differential equations has nonzero solutions. These lead us to eigenfunctions, which are vector solutions to the system of differential equations.

For normalization of \( v_1 \), we set \( c_1 = 1 \). This gives us \( v_1(0) = v_2(0) \), a case that we examine below in Section 4. We can also set \( c_1 = -1 \), which gives us \( v_1(0) = -v_2(0) \).

We can input this information into Mathematica to view visual representations of the eigenvalues and eigenfunctions. We use these graphs to determine where exactly the eigenvalues fall and can find their exact value.

We also want to examine eigenvalue curves. For example, the following graph shows us where the roots for a system occur in the places the curve crosses the x-axis.
This graph shows where the roots are located along a certain path for \( v_1 \) and \( v_2 \).

4 Symmetries

We will look at a few conditions that tell us about the symmetry of the system. For these instances we will let \( \xi = is, \ s \in \mathbb{R} \). Thus, when substituted into the Zakharov-Shabat system, we have:

\[
\begin{align*}
  v_1'(t) &= -sv_1 + q(t)v_2 \\
  v_2'(t) &= -q(t)v_1 - sv_2.
\end{align*}
\]

We will set up this equation so that \( v_1(-d) = 1, v_2(-d) = 0 \). Computing the solutions we get

\[
v(t) = \phi(t + d)v(-d),
\]

\[
v(t) = \begin{bmatrix}
  \cos \gamma (t + d) - \frac{i\xi}{\gamma} \sin \gamma (t + d) & \frac{h}{\gamma} \sin \gamma (t + d) \\
  -\frac{h}{\gamma} \sin \gamma (t + d) & \cos \gamma (t + d) + \frac{i\xi}{\gamma} \sin \gamma (t + d)
\end{bmatrix}
\begin{bmatrix}
  1 \\
  0
\end{bmatrix},
\]

\[
\begin{bmatrix}
v_1(t) \\
v_2(t)
\end{bmatrix} = \begin{bmatrix}
  \cos \gamma (t + d) - \frac{i\xi}{\gamma} \sin \gamma (t + d) \\
  -\frac{h}{\gamma} \sin \gamma (t + d)
\end{bmatrix}, \quad -d \leq t \leq d.
\]
We will assume that \( q(t) \) is even; that is, \( q(t) = q(-t) \). We claim that if \( v(t) \) is an eigenfunction for the eigenvalue \( \xi \) with \( \text{Im}(\xi) > 0 \), then \( v(t) \) must satisfy either of the following conditions:

1. \( v_1(0) = v_2(0) = 1 \)
2. \( v_1(0) = -v_2(0) = 1 \)

Only one of the above conditions can be satisfied at once because if \( v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} \) is an eigenfunction, then \( z(t) = \begin{pmatrix} v_2(-t) \\ v_1(-t) \end{pmatrix} \) is also an eigenfunction because of the following:

\[
\begin{align*}
z'_1 &= (v_2(-t))' = -v'_2(-t) \\
&= q(-t)v'_1(-t) - i\xi v_2(-t) \\
&= q(t)z_2(t) - i\xi z_1(t)
\end{align*}
\]

which is the first equation of the Zakharov-Shabat system. The second equation in the system is given by:

\[
\begin{align*}
z'_2 &= (v_1(-t))' = -v'_1(-t) \\
&= -(-i\xi v_1(-t) + q(-t)v_2(-t)) \\
&= i\xi z_2(t) - q(t)z_1(t).
\end{align*}
\]

So, if \( v \) is an eigenfunction, then any other linearly independent solution of the Zakharov-Shabat system is not decaying to zero at either \(+\infty\) or \(-\infty\). This solution would not satisfy the conditions of \( v_1(d) = 0 \) or \( v_2(-d) = 0 \) and would therefore exponentially increase and diverge as \( t \to +\infty \) or as \( t \to -\infty \), respectively. This implies that both \( v(t) \) and \( z(t) \) are multiples of each other. Therefore, there must be a constant \( c \neq 0 \) such that

\[
z(t) = cv(t).
\]

This gives us

\[
\begin{align*}
z_1(t) &= cv_1(t) = v_2(-t) \\
z_2(t) &= cv_2(t) = v_1(-t)
\end{align*}
\]

\[
\Rightarrow \quad cv_1(0) = v_2(0) \\
cv_2(0) = v_1(0)
\]

This gives us \( c = \pm 1 \). \( v_1(0) = 0 \) cannot be possible because it implies that \( v_2(0) = 0 \) and thus \( v(t) = 0 \) for all \( t \). Instead we have the following:

For \( c = 1 \): \( z(t) = v(t) \) \( \text{i.e.} \)

\[
\begin{align*}
v_2(-t) &= v_1(t) \\
v_1(-t) &= v_2(t) \\
v_2(0) &= v_1(0)
\end{align*}
\]

This gives us case a).
For $c = -1$: $z(t) = -v(t)$ \textit{i.e.} \[ v_2(-t) = -v_1(t) \]
\[ -v_1(-t) = v_2(t) \]
\[ v_2(0) = -v_1(0) \]

This gives us case \( b \). We will now look at these two scenarios.

\textbf{a)} First, we examine the case where \( v_1(0) = v_2(0) = 1 \) and \( v_1(d) = 0 \).

We let \( v(t) = \phi(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), where \( \phi(t) = \begin{bmatrix} \cos(\gamma t) - \frac{i\xi}{\gamma} \sin(\gamma t) & \frac{h}{\gamma} \sin(\gamma t) \\ -\frac{h}{\gamma} \sin(\gamma t) & \cos(\gamma t) + \frac{i\xi}{\gamma} \sin(\gamma t) \end{bmatrix} \)

which represents the value of \( c = -1 \).

So,

\[ v(t) = \begin{bmatrix} \cos(\gamma t) - \frac{i\xi}{\gamma} \sin(\gamma t) & \frac{h}{\gamma} \sin(\gamma t) \\ -\frac{h}{\gamma} \sin(\gamma t) & \cos(\gamma t) + \frac{i\xi}{\gamma} \sin(\gamma t) \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

We now plug in the parameter \( \xi = is \), which changes use \( \gamma = \sqrt{h^2 - s^2} \), and gives us the following result.

\[ v(t) = \begin{bmatrix} \cos(\gamma t) + \frac{\sin(\gamma t)}{\gamma} (s + h) \\ \cos(\gamma t) - \frac{\sin(\gamma t)}{\gamma} (s + h) \end{bmatrix} \]

Then,

\[ v(\gamma) = \begin{bmatrix} \cos(\gamma d) + \frac{\sin(\gamma d)}{\gamma} (s + h) \\ \cos(\gamma d) - \frac{\sin(\gamma d)}{\gamma} (s + h) \end{bmatrix} \]

We examine \( v_1(\gamma) = 0 \). This gives us the equation for the eigenvalues for this case:

\[ \cos \left( d\sqrt{h^2 - s^2} \right) + \frac{h + s}{\sqrt{h^2 - s^2}} \sin \left( d\sqrt{h^2 - s^2} \right) = 0. \]

We will use \( u = 2d\sqrt{h^2 - s^2} \) for \( 0 \leq u \leq 2dh \). Substituting \( u \) into the eigenvalue equation, we have

\[ -\frac{u}{2} \cot \left( \frac{u}{2} \right) = dh + \frac{1}{2} \sqrt{4d^2h^2 - u^2}. \]

\textbf{b)} In comparison, we will now look at the case where \( v_1(0) = -v_2(0) = 1 \).
This equation is similar to the one above except the final eigenvalue equation varies based on the initial conditions. So, instead of \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) we will use \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

\[
v(t) = \begin{bmatrix}
\cos(yt) - \frac{i\xi}{\gamma} \sin(yt) & \frac{h}{\gamma} \sin(yt) \\
-\frac{h}{\gamma} \sin(yt) & \cos(yt) + \frac{i\xi}{\gamma} \sin(yt)
\end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

We now plug in the parameter \( \xi = is \) to give us the following result.

\[
v(t) = \begin{bmatrix}
\cos(yt) + \frac{\sin(yt)}{\gamma} (s - h) \\
-\cos(yt) + \frac{\sin(yt)}{\gamma} (s - h)
\end{bmatrix}
\]

Then,

\[
v(d) = \begin{bmatrix}
\cos(yd) + \frac{\sin(yd)}{\gamma} (s - h) \\
-\cos(yd) + \frac{\sin(yd)}{\gamma} (s - h)
\end{bmatrix}.
\]

Thus, the equation for the eigenvalues for this case is:

\[
cos \left( d\sqrt{h^2 - s^2} \right) + \frac{-h + s}{\sqrt{h^2 - s^2}} \sin \left( d\sqrt{h^2 - s^2} \right) = 0.
\]

Using \( u \) from above, this gives us

\[-\frac{u}{2} \cot \left( \frac{u}{2} \right) = -dh + \frac{1}{2} \sqrt{4d^2h^2 - u^2}.\]

The following graph is an example the starting potential for a system.
From this, we examine where eigenvalues appear based off of the value for $\mu$. As seen below, an eigenvalue appears as the curve crosses the x-axis near point .33.

We also want to examine a case where there are two bumps that are separated from each other. In this situation all of the eigenvalues appear on the imaginary axis. We will be using the function $q_R(t)$.

For the instances of the graph with two separated bumps, we use the values of $0 < R < 2$.

The eigenvalues in this type of situation can be complex and non-imaginary. The two boundary conditions for this at $t = \frac{R}{2}$ are:

\[ v_1 \left( \frac{R}{2} \right) = v_1 \left( \frac{R}{2} \right), \]
\[ v_1 \left( \frac{R}{2} \right) = -v_2 \left( \frac{R}{2} \right). \]

The graph of the curve for this type of scenario in this system is shaped different from that in the case with only one bump. There are two curves in the line in this graph that show the way the eigenvalues are interacting and whether or not they collide.
In the following situation, the two rectangles are superposed. This scenario has eigenvalues that are purely imaginary. The line of symmetry is also not along the y-axis.

References
