THE TOPOLOGY ON THE SPACE OF LOCALLY INVARIANT ORDERS ON A GROUP

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Abstract. We show that the space of locally invariant orders on the group \( \mathbb{Z} \) has two non-isolated points, and all other points isolated.

1. Introduction

Recall that the set of all binary relations \( \mathcal{R} \) on a set \( S \) can be identified with \( 2^{S \times S} \), the set of all functions \( S \times S \to \{0, 1\} \), where given \( R \in \mathcal{R} \), we associate it with the function \( f_R \in 2^{S \times S} \) defined by \( f_R(a, b) = 1 \) if \( aRb \), and \( f_R(a, b) = 0 \) otherwise (here we let 2 denote \( \{0, 1\} \)). We give \( \{0, 1\} \) the discrete topology and \( 2^{S \times S} \) the product topology. It is well-known that \( 2^{S \times S} \) is a compact Hausdorff space, and is metrizable if \( S \) is countable.

We shall prove

**Theorem 1.1.** The space of locally invariant orders on the \( \mathbb{Z} \) has precisely two points which are not isolated. These two points are the usual order, and the reverse of the usual order. All other points are isolated.

Before fully discovering locally invariant orders on \( \mathbb{Z} \), I need to thank Dr. Linnell here who has made huge contribution by teaching and lightening me for this subject. Without his help, this paper would be pretty “empty”.

2. Process

2.1. Definition of Locally Invariant Order.

**Definition 2.1.** A locally invariant order (LIO) on a group \( G \) is a strict partial order \( < \) satisfying for all \( x, y \in G \) with \( y \neq e \), either \( x < xy \) or \( x < x^{-1}y \).

2.2. Explore the LIO on \( \mathbb{Z} \) by left-ordered group.

First of all, let us look at the right-ordered group and the left-ordered group, and try to find their relation with LIO.

A group \( G \) is left-ordered if it has a total order \( \leq \) which is left invariant, i.e., \( x \leq y \) implies that \( gx \leq gy \), \( \forall g, x, y \in G \). Similarly, a group \( G \) is right-ordered if it has a total order \( \leq \) which is left invariant, i.e., \( x \leq y \) implies that \( xg \leq yg \) for \( \forall g, x, y \in G \).

Prove: A group \( G \) is right-ordered if and only if \( G \) is left-ordered.

**Proof.** Let us define a concept of \( 1 \leq y \) such that \( x \leq y \iff x^{-1} \leq y^{-1} \).

Assume a group \( G \) is right-ordered, so let \( x \leq y \), we can get \( x^{-1}y^{-1} \leq y^{-1}y^{-1} \) by assumption. That is \( (gx)^{-1} \leq (gy)^{-1} \) by the property of inverse element in a

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group. Then, again, by applying the definition of \( \leq_1 \), we acquire \( gx \leq_1 gy \). So, we get \( G \) is also left-ordered. By using the similar method, we can prove the inverse order.

Next, we want to show that a left-ordered group is torsion free.

In abstract algebra, a group is torsion-free if the only element of finite order is the identity. 

Prove: A left-ordered group is torsion-free.

Proof. The proof is by contradiction, assume \( G \) has an element of finite order such that \( a^n = e_G \) for \( a \in G \), where \( e_G \) is the identity of the group \( G \). Hence, we need to deal with two cases.

1. \( e_G < a \), then \( ae_G < aa \) by left-ordered property, so \( a < a^2 \) by the identity. 
   so \( e_G < a < a^2 < a^3 < \cdots < a^n = e_G \), so we get \( e_G < e_G \), we reached a contradiction.

2. \( e_G > a \), similarly, we can get \( e_G = a^n < a^{n-1} < \cdots < a < e_G \), so we reached another contradiction.

By combining 1 and 2, we can say a left-order group is torsion-free.

We have explored the properties about the left-ordered group and the right-ordered group. Does a left-ordered group or a right-ordered group have a locally invariant order?

Prove: A left-ordered group has a locally invariant order.

Proof. The proof is by contradiction, assume the left-ordered group does not have a locally invariant order, i.e., \( \exists x, y \in G \), with \( y \neq e \), such that \( x \geq xy \) and \( x \geq xy^{-1} \). 

By \( xy \leq x \), we can multiply both sides by \( x^{-1} \), so we can get \( x^{-1}xy \leq x^{-1}x \) due to the property of left-ordered group, hence we obtain \( y \leq e \). By \( xy^{-1} \leq x \), we can get \( y^{-1} \leq e \) by same method, then we times \( y \) by both sides using the property of the left-ordered group, we get \( e \leq y \). Combining these two, we can easily obtain that \( y \) is forced to be \( e \), which contradicts our assumption.

So, a left-ordered group has a locally invariant order.

2.3. Find all LIO on \( \mathbb{Z} \).

From section 2.2, we have proven that a left-ordered group has a locally invariant order. We want to know whether the inverse can work as well, i.e, try to find a LIO on \( \mathbb{Z} \), which is a left order. This statement is trivial if we choose our LIO to be usual order, which is like \( \cdots < -2 < -1 < 0 < 1 < 2 < \cdots \) on \( \mathbb{Z} \). Hence, we want to know are all LIO on \( \mathbb{Z} \) left-orders?

Find a LIO on \( \mathbb{Z} \) (the group of integers under the operation addition), which is not a left order. (Right now, we really need to figure what LIO really means. The definition of LIO is “a locally invariant order (LIO) on a group \( G \) is a strict partial order \( \prec \) satisfying for all \( x, y \in G \) with \( y \neq e \), either \( x < xy \) or \( x < xy^{-1} \).” For a given group, like \( \mathbb{Z} \), it only gives us the “number” elements in this group, there is no comparison between elements unless we define one. LIO is a useful tool for us to define “big and small” inside a group. For example, \( 1 < 2 \) is correct under the usual order, which we are familiar with. However, we can also say \( 2 < 1 \) is also correct if we are talking about the opposite order, i.e., \( \cdots < 2 < 1 < 0 < -1 < -2 < \cdots \))
Solution: Let us define an order on $\mathbb{Z}$ satisfying the following property

$$\begin{cases} 0 < 1 < 2 < 3 < 4 < \cdots \\ 0 < -1 < -2 < -3 < -4 < \cdots \end{cases}$$

(This is a LIO on $\mathbb{Z}$, which shall be proven later.)

Choose $x = 1$, $y = 2$, $g = -3$, such that $x = 1 \leq 2 = y$ and $g + x = -2$, $g + y = -1$, so that $g + x \geq g + y$, which is not left-ordered.

Now, let us prove this indeed a LIO on $\mathbb{Z}$

Prove: On $\mathbb{Z}$, which satisfies

$$\begin{cases} 0 < 1 < 2 < 3 < 4 < \cdots (\mathbb{Z}^+) \\ 0 < -1 < -2 < -3 < -4 < \cdots (\mathbb{Z}^-) \end{cases}$$

is a LIO.

Proof. In order to prove this is a LIO on $\mathbb{Z}$, it is better for us to let the elements in the first line to be $\mathbb{Z}^+$, and elements in the second line to be $\mathbb{Z}^-$. Hence, by the definition of locally invariant order, we need four cases to solve it.

1. if $x, y \in (\mathbb{O})$, then $x < x + y$.
2. if $x \in (\mathbb{O})$, then $x < x + y$.
3. if $x \in (\mathbb{O})$, then $y^{-1} \in \mathbb{Z}^+$, so it satisfies that $x < x + y^{-1}$.
4. if $x \in (\mathbb{O})$, then we have $y^{-1} \in \mathbb{Z}^-$, so it satisfies that $x < x + y^{-1}$.

Due to four cases, we can conclude that this is a LIO on $\mathbb{Z}$ (look at figure 1).

It is not very difficult to find one LIO on $\mathbb{Z}$ as we did above, which is not a left order. Can we try to find more LIOs on $\mathbb{Z}$?

Purpose: Try to find more LIO on $\mathbb{Z}$ (the group of integers under the operation addition). First, we can separate $\mathbb{Z}$ into even integers part and odd integers part, and try to prove it by using definition.

$T_1$: Try odd and even subsets: Under $\mathbb{Z}$, let us create this:

$$\begin{cases} \cdots < -3 < -1 < 1 < 3 < 5 < \cdots (\mathbb{O}) \\ \cdots < -2 < 0 < 2 < 4 < 6 < \cdots (\mathbb{E}) \end{cases}$$

1. if $x, y \in (\mathbb{O})$, then $x < x + y$.
2. if $x \in (\mathbb{O})$, then $x < x + y$.
3. if $x \in (\mathbb{O})$, $y \in (\mathbb{O})$, where $y \neq 0$, then $x + y \in (\mathbb{O})$, it leads to no comparison between $x$ and $x + y$ since one is odd number, and another is even number, which is not defined. Hence, this is not a LIO.

So, this is not a LIO.

By trying the example of even and odd subsets, we can clearly see that LIO does not have a property of “jump” between one certain category to another. Next, since we have done the case, which we choose 0 as our “bending” point (proved), why cannot we try -1 this time, and see how it goes?

$T_2$: Try to choose -1 as “bending” point, so let us create this:

$$\begin{cases} -1 < 0 < 1 < 2 < 3 < \cdots (1) \\ -1 < -2 < -3 < -4 < \cdots (2) \end{cases}$$

This is slightly tougher to prove, since we need to deal with -1 very carefully.
Figure 1. A left flip at 0

(1) if \( x, y \in (1) \), then we need few cases
   - if \( x, y \in \mathbb{Z}^+ \), then \( x < x + y \) with \( y \neq 0 \).
   - if \( x \in \mathbb{Z}^+ \), \( y = -1 \), then \( x < x + y \).
   - if \( x = 0 \), \( y = -1 \), we have \( y^{-1} = 1 \), so \( x < x + y^{-1} \).

(2) if \( x, y \in (2) \), then \( x < x + y \) obviously.

(3) if \( x \in (1) \) and \( y \in (2) \), then \( y^{-1} \in (1) \), so \( x < x + y^{-1} \) must be true.

(4) if \( x \in (2) \) and \( y \in (1) \), then \( y^{-1} \in (2) \), so \( x < x + y^{-1} \) must be true.

Hence, by bending at \(-1\), this is a LIO on \( \mathbb{Z} \) (Similarly, as figure 2 shows).

It is very easy to prove this LIO is not a left order. Choose \( x = -1 \), \( y = -2 \), \( g = 3 \), then \( g + x = 2 \), and \( g + y = 1 \), so we get \( g + x > g + y \). So, it is not a left-ordered group.

Based on choosing the bending point at 0 and \(-1\), which all work well, we can come to an assumption.

Assumption: If we bend or flip the integer line at any integer point, it always satisfies the property of LIO, and will not be a left order. We will prove this assumption informally and formally later. However, all LIOs we have found are all left flipped. For example,

\[
\begin{align*}
0 &< 1 < 2 < 3 < 4 < \cdots \\
0 &< -1 < -2 < -3 < -4 < \cdots
\end{align*}
\]
Figure 2. A left flip at −1

is a left flip order at point “0”. Basically, we list all elements of \( \mathbb{Z} \) in usual order, like \( \cdots < -2 < -1 < 0 < 1 < 2 < \cdots \), and we keep the relations of the elements after 0 unchanged, and we flip the relations of the elements before 0. This will lead us the example.

Define left flip order formally:

**Definition 2.2.** Let \( n \in \mathbb{Z} \), a left flip order \( \prec \) at \( n \) is an order which satisfies \( n \prec m \) if \( n < m \), and \( n \prec m \) if \( m < n \), where \( < \) stands for usual order.

Last but not least, we have shown that a left flip order is a LIO on \( \mathbb{Z} \) and not a left-ordered group. In this case, we want to show a right flip order is also a LIO and not a left order as well. Choose this right order as our specific example:

\[
\begin{align*}
1 & > 2 > 3 > 4 > \cdots (1) \\
1 & > 0 > -1 > -2 > \cdots (2)
\end{align*}
\]

However, it turns out “right flip order” is not a LIO as we expected above. If we choose our \( x \) to be 2, and \( y \) to be 5, we know by definition 2.1 that either \( x < xy \) or \( x < xy^{-1} \) should be true. In this case, \( x + y = 7 \), and \( x = 2 \neq 7 \), so first condition fails. Moreover, \( x + y^{-1} = -3 \), which has no comparison with \( x = 2 \) since not defined by our example.
Similarly, let us expand our example to general case. If we “right flip” the integer line at any point, which means keep the relation before bending point unchanged, and flip the relation of the elements after this bending point, this will NOT lead to any LIO as our example showed up.

At this section, we have shown:

1. A left-ordered group has a LIO.
2. A LIO does not have to be a left-ordered group.
3. A LIO does not allow any “jump”, which has been shown as our example of odd and even subsets.
4. A left flip order has a very “high chance” to be a LIO.

By combing all these, we can come to an assumption that LIO is either left-ordered or left-flipped.

2.4. **Prove a LIO is either left-ordered or left-flipped.**

Prove: On \( \mathbb{Z} \), a LIO is either left-ordered or left-flipped (informally).

*Proof.* By definition 2.1, we start from specific case.

\[
\begin{cases}
\text{either } 0 < 0 + 1 \\
\text{or } 0 < 0 - 1
\end{cases}
\]

which gives us 2 cases generally.

Assume \( 0 < -1 \), similarly, we follow definition 2.1, we have either \(-1 < -1 + 1 \) or \(-1 < -1 - 1 \) (by going to the “right” direction). However, \(-1 < -1 + 1 \iff -1 < 0 \), which violates our first assumption, so \(-1 < -1 - 1 \iff -1 < -2 \) must be true. Hence, by following the same steps for infinitely many times, we can obtain this easily: \( 0 < -1 < -2 < -3 < -4 < \cdots (\ast) \). By going to “right” direction is pretty straightforward, we need to deal with the “left” direction next. The “left” direction leads to two cases at the beginning, we have either \( 1 < 1 + 1 \iff 1 < 2 \) or \( 1 < 1 - 1 \iff 1 < 0 \) is true.

Suppose \( 1 < 2 \), we obtain either \( 2 < 2 + 1 \) or \( 2 < 2 - 1 \) is true. However, \( 2 < 2 - 1 \iff 2 < 1 \), which violates the assumption of \( 1 < 2 \), so \( 2 < 2 + 1 \iff 2 < 3 \) must be true. so by following the same logic, we can get \( 1 < 2 < 3 < 4 < 5 < \cdots \). By combining with (\( \ast \)), we can obtain these relations:

\[
\begin{cases}
1 < 2 < 3 < 4 < 5 < \cdots \\
0 < -1 < -2 < -3 < -4 < \cdots
\end{cases}
\]

That is for \( 1 < 2 \), what about the case of \( 1 < 0 \)?

Suppose \( 1 < 0 \), this leads to two small sub cases again, we have either \( 2 < 2 + 1 \iff 2 < 3 \) or \( 2 < 2 - 1 \iff 2 < 1 \) is true. By looking at \( 2 < 3 \), we have either \( 3 < 4 \) or \( 3 < 2 \) is true, in this case \( 3 < 4 \) is our only option here since \( 3 < 2 \) contradicts \( 2 < 3 \), by following the same logic, we can get \( 2 < 3 < 4 < 5 < \cdots \). By combining with (\( \ast \)), we get these relations:

\[
\begin{cases}
2 < 3 < 4 < 5 < 6 < 7 < \cdots \\
1 < 0 < -1 < -2 < -3 < -4 < \cdots
\end{cases}
\]

After finishing the case \( 2 < 3 \), we need to take a look at the case \( 2 < 1 \). Similarly, by going just one step to the “left”, we can get either \( 3 < 4 \) or \( 3 < 2 \) is true. \( 3 < 4 \) can bring us this following relations:

\[
\begin{cases}
3 < 4 < 5 < 6 < 7 < 8 < \cdots \\
2 < 1 < 0 < -1 < -2 < -3 < \cdots
\end{cases}
\]
After finishing the case \( 3 < 4 \), we need to look at the case \( 3 < 2 \). By going just one step to the “left” again, which generates two sub cases again. Following the same logic, if we do this step for infinitely many times, we can acquire these relations:

\[
\{ \emptyset \ldots < 4 < 3 < 2 < 1 < 0 < -1 < -2 < -3 < -4 < \ldots \}
\]

This is the opposite order on \( \mathbb{Z} \), which is just the reverse of the usual order: \( \ldots < -4 < -3 < -2 < -1 < 0 < 1 < 2 < 3 < 4 < \ldots \), and this is left-ordered \( \) (figure 3).

Since the case \( 0 < -1 \) can bring us one opposite order, and infinitely many “left space flip order”, the case \( 0 < 1 \) shall bring us one usual order, and infinitely many “left space flip order” as well. If we shift our starting point from 0 to any integer \( n \in \mathbb{Z} \), the proof is done here. \( \square \)

In the proof, we mentioned a term “left space flip order”, which is little different from the term we have known above: “left flip order”. As we can easily tell the difference between these two orders, for example:

\[
\{ \begin{array}{c}
3 < 4 < 5 < 6 < 7 < 8 < \ldots \\
2 < 1 < 0 < -1 < -2 < -3 < \ldots
\end{array}
\]

we cannot compare 2 with 3 for this case. There is a “break” point between 2 and 3, and that is why we call it “space flip” \( \) (figure 4). On the other hand, “left flip order” does not have this issue here, for example:

\[
\{ \begin{array}{c}
0 < 1 < 2 < 3 < 4 < \ldots \\
0 < -1 < -2 < -3 < -4 < \ldots
\end{array}
\]

Both lines start from point 0. Hence, if we just add one more condition to connect 2 and 3 in this case, “left space flip order” would become “left flip order”. Let’s say \( 2 < 3 \), it gives us

\[
\{ \begin{array}{c}
2 < 3 < 4 < 5 < 6 < 7 < 8 < \ldots \\
2 < 1 < 0 < -1 < -2 < -3 < \ldots
\end{array}
\]

which is a left flip order obviously.

Let us give the definition of “left space flip order” or “LSF” here:

**Definition 2.3.** Let \( n \in \mathbb{Z} \), a left space flip order \( \prec \) at \( n \) is an order which satisfies \( n < m \) if \( n < m \), and \( n - 1 < m \) if \( m < n - 1 \), and neither \( n \prec n - 1 \) nor \( n - 1 \prec n \) is true, where \( \prec \) stands for usual order.

Let us prove the left space flip order is a locally invariant order formally. At this time, we should define a new term to help us.

**Definition 2.4.** Let \( N \) be the smallest integer such that \( N < N + 1 \), that is if an integer \( a \) is bigger than \( N \), then \( a < a + 1 \), but if \( a \) is smaller than \( N \), then \( a \not< a + 1 \).

Prove: The left space flip order is a locally invariant order.
Figure 4. Left space flip order at 2 and 3

Proof. By definition 2.3, we need to make a comparison between \( n \) and \( N \) here. If \( n \) is at least \( N \), then \( n \) satisfies the property of \( N \), so we have \( n \prec n + 1 \prec n + 2 \prec n + 3 \prec \cdots \prec n + y \), where \( y \) is a non-identity element of \( \mathbb{Z} \), so by definition 2.1, it is a LIO. If \( n \) at most \( N - 1 \), then \( n \leq N \), so \( n \prec n - 1 \prec n - 2 \prec \cdots \prec n - y \), so it is also a LIO by definition 2.1. Hence, the left space flip order or LSF is a LIO. \( \square \)

Since definition 2.4 is so convenient and useful, we can use this new definition to reprove: a LIO is either the LSF/LF order or the usual/opposite order formally this time. Prove: LIO is either the left space flip/left flip order or the usual/opposite (left-ordered) order.

Proof. By the choices of \( N \), this leads to 3 cases.
Theorem 2.6. Let $X$ be the product of the topological spaces $X_\beta$ and let $S_\beta$ denote the collection $S_\beta = \{\pi^{-1}_\beta(U_\beta) | U_\beta \text{ open in } X_\beta\}$ and let $S$ denote the union of these collections, $S = \bigcup_{\beta \in J} S_\beta$. The topology generated by the subbasis $S$ is called the product topology. [1]
so \( Q(a,b) = \cdots \{0,1\} \times \{0,1\} \times \{0\} \times \{0,1\} \times \{0,1\} \cdots \) is open. Hence, by the definition of 2.5, we get \( P(a,b) \) is closed. Similarly, we can obtain that \( P(b,c), P(a,c), Q(a,b) \) and \( Q(b,c) \) are all closed. By the following theorem:

**Theorem 2.7.** Let \( X \) be a topological space. Then the following conditions hold:

1. \( \emptyset \) and \( X \) are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed. [1]

We can get \( K(a,b,c) = Q(a,b) \cup Q(b,c) \cup (P(a,b) \cap P(b,c) \cap P(a,c)) \) is closed. Moreover, \( \bigcap_{(a,b,c)} K(a,b,c) \) is also closed due to number (2) from theorem 2.7. Since \( < \) is a partial order if and only if \( \prec \in \bigcap_{(a,b,c)} K(a,b,c) \), and \( \prec \) stands for a strict partial order, so strict partial orders form a closed subspace of \( 2^{\mathbb{Z} \times \mathbb{Z}} \).

According to the definition of LIO (2.1), it is not difficult to see that LIO is a strict partial order with certain property. What does this certain property means in terms of \( P(a,b) \) or \( 2^{\mathbb{Z} \times \mathbb{Z}} \)?

Let us fix \( a, g \in \mathbb{Z} \) with \( g \neq \text{identity} \), we have either \( a < a\,g \) or \( a < a\,g^{-1} \), so for \( a < a\,g \), it is all sequences which are 1 at \((a, a\,g)\) and arbitrary elsewhere, denoted as \( P(a,a\,g) \). And for \( a < a\,g^{-1} \), it is all sequences which are 1 at \((a, a\,g^{-1})\) and arbitrary elsewhere, denoted as \( P(a,a\,g^{-1}) \). Due to the logic, which is “either...or...”, \( \prec \in P(a,a\,g) \cup P(a,a\,g^{-1}) \). Next, in order to acquire the general case or arbitrary \( a \)
and $g$, we need $\prec \in \bigcap_{(a,g)}[P_{(a,ag)} \cup P_{(a,ag^{-1})}]$. Since LIO has the properties of strict partial order, we can conclude that $\prec \in \bigcap_{(a,g,b,c)}\{[P_{(a,ag)} \cup P_{(a,ag^{-1})}] \cap K_{(a,b,c)}\}$.

2.6. Isolated points on LIO.

First thing first, since we are going to focus on isolated points on LIO, we need to give the definition of isolated point on a topological space.

**Definition 2.8.** A point $P$ in a topological space $T$ is isolated means there exists an open set $O$ which contains $P$, but no other point of $T$. [1]

As we have explored and proven above, a LIO is either the left space flip order/left flip order or the usual/opposite order. We need to pay attention here, the isolated points we are trying to seek here are elements from LIO since we are dealing with relations on $\mathbb{Z}$ (LIO). Our final goal is to prove theorem 1.1, which includes two parts.

**Prove:**

1. the left space flip orders and the left flip orders are isolated.
2. the usual order and the opposite order are not isolated.

Here is the proof process for “every left flip order is isolated”.

**Proof.** Let $\prec_n$ be a flip order at $n \in \mathbb{Z}$, where $n$ is arbitrary. Choose our open set to be $P_{(n,n+1)}$ and $P_{(n,n-1)}$, such that all sequences which are 1 at $(n,n+1)$ and arbitrary elsewhere and all sequences which are 1 at $(n,n-1)$ and arbitrary elsewhere. Since $\prec_n \in (P_{(n,n+1)} \cap P_{(n,n-1)})$ and only, so $\prec_n$ is isolated. Hence, the left flip orders are isolated. 

By using the same method, we can also show that the left space flip orders are isolated.

It is kind of easy to show that: $\prec_n \in (P_{(n,n+1)} \cap P_{(n,n-1)})$ and only. Let us assume $(P_{(n,n+1)} \cap P_{(n,n-1)})$ contains another flip order, say $\prec_k$, where $k \neq n$.

1. if $k < n$, then $k \not< k + 1$, so $k + 1 < k$ by definition 2.4. Since $\prec_k \in (P_{(n,n+1)} \cap P_{(n,n-1)})$ by assumption, so

\[
\begin{align*}
&\begin{cases}
n < n+1 < n+2 < \cdots \\
n < n-1 < n-2 < \cdots 
\end{cases} \\
k < n & \text{ does not exist, so } k \text{ doesn’t exist.}
\end{align*}
\]

2. if $n < k$, then $k < k + 1$ by definition 2.4. Since $\prec_n \in (P_{(n,n+1)} \cap P_{(n,n-1)})$ and $\prec_k \in (P_{(k,k+1)} \cap P_{(k,k-1)})$, we have

\[
\begin{align*}
&\begin{cases}
n < n+1 < n+2 < \cdots \\
n < n-1 < n-2 < \cdots 
\end{cases} \\
k < k & \text{ does not exist, so } k \text{ doesn’t exist.}
\end{align*}
\]

by our assumption of $k$, say if $k = n + 1$, then $k - 1 = n$. However, we need to satisfy $k < k - 1 \iff n + 1 < n$, and haven $< n + 1$ at the same time, so it comes to a contradiction. Similarly, if $k = n + t$, where $t$ is arbitrary integer number, that will also lead to contradiction. So $k$ doesn’t exist. So $\prec_n \in (P_{(n,n+1)} \cap P_{(n,n-1)})$ and only.
Finally, we need to prove: the usual order and the opposite order are not isolated. We want to show the usual order is not isolated, which can bring us the opposite order is not isolated by using the same method. We want to find an open set $O$, which contains $<$ and $\prec$ at the same time, where we denote $<$ as the usual order and $\prec$ as our left flip order.

**Proof.** Let $\prec \in O$, where $O = (P_{(a_1, b_1)} \cap P_{(a_2, b_2)}) \cap \cdots \cap P_{(a_n, b_n)} \cap (Q_{(c_1, d_1)} \cap Q_{(c_2, d_2)} \cap \cdots \cap Q_{(c_m, d_m)})$ and we shall show $\prec \in O$, so we have $a_i < b_i$ and $d_i \leq c_i$ for all $i$. Choose $M = \min(a_i, d_i)$. If $a_i < d_i$, so $a_i = M$. By $\prec M$, we have

\[
\begin{align*}
M &< M - 1 < M - 2 < \cdots \\
M &< M + 1 < M + 2 < \cdots
\end{align*}
\]

by definition 2.3. Since $<$ is the usual order, so $a_i = M < M + 1 < M + 2 < M + 3 < \cdots$, where $a_i < b_i$, $a_i < d_i \leq c_i$ for all $i$. Since $\prec M$ satisfies all properties of $O$, so $\prec M \in O$, so generally, $\prec \in O$, so in this case, the usual order is not isolated. Similarly, if $d_i < a_i$, so $d_i = M$, which should give us the same result. □

Hence, theorem 1.1 has been proven.

### 3. Conclusion

As we have explored above, a locally invariant order on the group $\mathbb{Z}$ is either a left space flip/left flip order or the usual/opposite order, where here I use $\prec$ to represent the LSF order, and $<$ to represent the usual order. Moreover, LIO is a strict partial order with certain property, which can form a closed subspace of $2^{2\mathbb{Z} \times \mathbb{Z}}$. Last but not least, on the group $\mathbb{Z}$, every left space flip order or every left flip order is isolated, and the usual order and the opposite order are not isolated for LIO. There is more to say if we choose to expand our group from $\mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$ or from $\mathbb{Z}$ to $\mathbb{R}$, which shall be our next goal.

### References