Model Reduction of First Order State Space Systems 
by Rational Interpolation

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1 Statement of Problem

In this paper, we investigate approximations to the first order state space equation

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

(1.1)

where \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \) and \( D \in \mathbb{R}^{p \times m} \) are constant matrices, and \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), and \( y(t) \in \mathbb{R}^p \) are called the state variable, input and output of the system. The length of the state variable is denoted \( n \) and is called the dimension of the system. It will be our desire to form a system that approximates the original system with a reduced dimension. We will assume \( x(0) = 0 \). In the case where \( m = p = 1 \), the system is called a single-input/single-output (SISO) system, and a multi-input/multi-output (MIMO) system otherwise.

Solving systems of the form 1.1 can be very computationally expensive when the dimension \( n \) is of a high order. Thus we look for a system of a reduced dimension of \( r \) such that the output, \( y_r(t) \), is a good approximation for \( y(t) \). We will form this system as

\[
\begin{align*}
E_r \dot{x}_r(t) &= A_r x_r(t) + B_r u(t) \\
y_r(t) &= C_r x_r(t) + D_r u(t)
\end{align*}
\]

(1.2)

where \( E_r, A_r \in \mathbb{R}^{r \times r} \), \( B_r \in \mathbb{R}^{r \times m} \), \( C_r \in \mathbb{R}^{p \times r} \) and \( D_r \in \mathbb{R}^{p \times m} \). In order to quantify how good of an approximation the reduced system is, we need a way of measuring the error between the two, which is the next topic.

2 Error

Although it is most intuitive to look at the difference between \( y \) and \( y_r \) as the error, the frequency space will prove to be a very powerful tool. Let \( \hat{y}(s) \) denote the Laplace transform of \( y(t) \), and like wise for other terms. Then by taking the Laplace transform of both sides of 1.1 we obtain

\[
E(s\hat{x}(s)) = A\hat{x}(s) + B\hat{u}(s)
\]

(2.3)

and thus

\[
\hat{x}(s) = (sE - A)^{-1}B\hat{u}(s)
\]

(2.4)
We also have
\[ \hat{y}(s) = C\hat{x}(s) + D\hat{u}(s) \]  
and thus
\[ \hat{y}(s) = (C(sE - A)^{-1}B + D)\hat{u}(s) \]
This leads to the transfer functions for the original and reduced systems
\[ H(s) = C(sE - A)^{-1}B + D \]  
and
\[ H_r(s) = C_r(sE_r - A_r)^{-1}B_r + D_r \]
Both \( H(s) \) and \( H_r(s) \) are \( p \times m \) matrix-value rational functions, of degree \( n \) and degree \( r \) respectively. Note that
\[ \hat{y}(s) - \hat{y}_r(s) = (H(s) - H_r(s))\hat{u}(s) \]
which lead us to investigate the error between transfer functions as a measure of the error between the original and reduced system. The two most common choices for measuring the error between the transfer functions are \( \mathcal{H}_\infty \) norm and \( \mathcal{H}_2 \) norm.

2.1 The \( \mathcal{H}_\infty \) norm
The \( \mathcal{H}_\infty \) norm of a transfer function \( H(s) \) is defined as
\[ \|H(s)\|_{\mathcal{H}_\infty} = \sup_{w \in \mathbb{R}} \|H(iw)\|_2 \]  
where \( \|M\|_2 \) is the 2-induced norm of complex matrix \( M \). Minimizing the \( \mathcal{H}_\infty \) norm of the transfer function has the physical significance of directly minimizing the \( L_2 \) norm of the output \( y(t) \) in the time domain, as
\[ \|y - y_r\|_{L_2} \leq \|H - H_r\|_{\mathcal{H}_\infty} \|u\|_{L_2}. \]  
Thus when the goal is to minimize the \( L_2 \) norm of the output error, one should look to minimize the \( \mathcal{H}_\infty \) norm of the error of the transfer functions.

2.2 The \( \mathcal{H}_2 \) norm
The \( \mathcal{H}_2 \) norm of a transfer function \( H(s) \) is defined and
\[ \|H(s)\|_{\mathcal{H}_2} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \|H(iw)\|_F^2 \right)^{1/2} \]  
where \( \|M\|_F \) is the Frobenius norm of the complex matrix \( M \), which can be calculated as
\[ \|M\|_F = \text{trace}(M^*M). \]
For the $\mathcal{H}_2$ norm of $H$ to be finite, we require the eigenvalues of the matrix pencil $\lambda E - A$ to lie in the left half plane and $D = 0$. Like the $\mathcal{H}_\infty$ norm, minimizing the error between transfer functions with respect to the $\mathcal{H}_2$ leads to minimizing the error between the outputs, in this case, with respect to the $L_\infty$ norm. Indeed we have

$$\|y - y_r\|_{L_\infty} \leq \|H - H_r\|_{\mathcal{H}_2}\|u\|_{L_2}. \quad (2.14)$$

Thus when wanting to minimize the error between the outputs with respect to $L_\infty$ norm, one should look to minimize the error between the transfer function with respect to the $\mathcal{H}_2$ norm, which will be our goal.

### 3 The Reduced System

The state variable $x(t)$ evolves in a $n$ dimensional subspace, but we assume there exists a $r$ dimensional subspace for which $x(t)$ lies close to. In order to form our reduced system in the form of 1.2, we let $V \in \mathbb{R}^{n \times r}$ be a basis for this $r$ dimensional subspace, thus we have $x(t) \approx VX_r(t)$, where $X_r(t)$ will be our reduced system state variable. We then obtain the residue

$$R(X_r(t)) = EV\dot{X}_r(t) - AX_r(t) - Bu(t) \quad (3.15)$$

and an approximate output $y_r(t)$

$$y_r(t) = CVX_r(t) + Du(t) \quad (3.16)$$

Then by enforcing a Petrov-Galerkin orthogonality condition on the residual, we obtain the expressions for the reduced matrices. We assume there exists a basis $W \in \mathbb{R}^{n \times r}$ such that

$$W^T R(X_r(t)) = W^T (EV\dot{X}_r(t) - AVX_r(t) - Bu(t)) = 0 \quad (3.17)$$

and thus

$$W^T EV\dot{X}_r(t) = W^T AVX_r(t) + W^T Bu(t) \quad (3.18)$$

this along with 3.16, gives us 1.2 when we define

$$E_r = W^T EV, \ A_r = W^T AV \\quad B_r = W^T B, \ C_r = CV, \ D_r = D \quad (3.19)$$

It is important to note the particular bases $V$ and $W$ are not unique to a reduced model. Let $T_1$ and $T_2$ be nonsingular matrices in $\mathbb{R}^{r \times r}$ corresponding to a change in basis, and let $V = VT_1$ and $W = WT_2$ be $V$ and $W$ in those bases. This leads to a new set of reduced matrices defined as $\tilde{E}_r = T_2^T E_r T_1$, $\tilde{A}_r = T_2^T A_r T_1$, $\tilde{B}_r = T_2^T B_r$, $\tilde{C}_r = C_r T_1$ and
Our goal is to find a degree-4 rational $p \times m$ matrix valued function $H_r(s)$ that approximates the degree-$n$ $p \times m$ matrix valued function $H(s)$ in the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norms, thus approximating the output of the system $y$ in the $L_2$ or $L_\infty$ norms. One tool that can be used is rational interpolation, but to achieving full matrix interpolation, the degree-$r$ rational $p \times m$ matrix valued function $H(s)$ at $s = \sigma$ along the right tangential direction vector $r \in \mathbb{C}^m$ if

$$H(\sigma)r = H_r(\sigma)r$$

By similar definition, $H_r(s)$ is a left tangential interpolant to $H(s)$ at $s = \mu$ along the left tangential direction vector $l \in \mathbb{C}^p$ if

$$l^T H(\mu) = l^T H_r(\mu)$$

We call $H_r(s)$ a bi-tangential Hermite interpolant to $H(s)$ at $s = \sigma$ along the right direction vector $r \in \mathbb{C}^m$ and left direction vector $l \in \mathbb{C}^p$ when

$$l^T H'(\sigma)r = l^T H'_r(\sigma)r$$

where $H'(\sigma)$ is the derivation of $H$ with respect to $s$, at $s = \sigma$. In order to satisfy this last condition, we will enforce our right and left interpolation point to be equal, and we look for a $H_r$ that satisfies

$$H(\sigma_i)r_i = H_r(\sigma_i)r_i$$

$$l_i^T H(\sigma_i) = l_i^T H_r(\sigma_i)$$

$$l_i^T H'(\sigma_i)r_i = l_i^T H'_r(\sigma_i)r_i$$

(4.21)

for a set of $r$ interpolation points $\{\sigma_i\}_{i=1}^r \in \mathbb{C}$, $r$ right directional vectors $\{r_i\}_{i=1}^r \in \mathbb{C}^m$, and $r$ left directional vectors $\{l_i\}_{i=1}^r \in \mathbb{C}^p$. Now our goal becomes to find the bases $V$ and $W$ such that the reduced transfer function $H_r(s)$ obtained as in 2.8 satisfies the conditions of

$$\hat{D}_r = D_r = D.$$
4.21. We can guarantee this when we enforce

\[(\sigma_i E - A)^{-1} B r_i \in \text{Ran}(V)\]

and

\[
\left( I^T C (\sigma_i E - A)^{-1} \right)^T \in \text{Ran}(W).
\]

which guarantees \( H(\sigma_i) r_i = H_r(\sigma_i) r_i \) and \( l_i^T H(\sigma_i) = l_i^T H_r(\sigma_i) \) respectively. And we obtain \( l_i^T H'(\sigma_i) r_i = l_i^T H'_r(\sigma_i) r_i \) for free, just by having both of the previous conditions hold. This assumes that \( \sigma E - A \) is invertible and that the right and left directional vectors are nontrivial.

We wish to find the a reduced order transfer function \( H_r(s) \) that minimizes the error in the \( \mathcal{H}_2 \) norm, thus

\[
\|H - H_r\|_{\mathcal{H}_2} = \min_{\dim(H_r) = r} \|H - H_r\|_{\mathcal{H}_2}
\]

as this minimizes the error between the system outputs in the \( L_\infty \) norm. We will assume that \( E \) is nonsingular and thus \( \lim_{s \to \infty} H(s) = D \). Thus to have bound error \( \|H - H_r\|_{\mathcal{H}_2} \), we need \( D_r = D \). We will assume without lose of generality that \( D_r = D = 0 \). For MIMO systems, the conditions for optimizing the \( \mathcal{H}_2 \) error given in 4.21 can be analyzed in the pole-residue expansion for \( H_r(s) \). We look to write the reduced transfer function as

\[
H_r(s) = \sum_{i=1}^{r} \frac{l_i r_i^T}{s - \lambda_i}
\]

(4.22)

The \( \lambda_i \)'s are known as the poles of \( H_r(s) \) and we assume these to be distinct. Also \( l_i r_i^T \) is the residue of \( H_r(s) \) at \( s = \lambda_i \). The derivation of the poles and residues is now shown. We start with the expression for the reduced system 1.2 and define \( \tilde{A}_r = E^{-1} A_r, \tilde{B}_r = E^{-1} B_r \) to yield

\[
\dot{x}_r(t) = \tilde{A}_r x_r(t) + \tilde{B}_r u(t)
\]

(4.23)

which yields the transfer function

\[
H_r = C_r (sI - \tilde{A}_r)^{-1} \tilde{B}_r
\]

(4.24)

Recall we let \( D_r = 0 \). We then assume \( \tilde{A}_r \) has an eigenvalue decomposition of \( \tilde{A}_r = V \Lambda V^{-1} \) and thus

\[
H_r = C_r (sV V^{-1} - \Lambda V^{-1})^{-1} \tilde{B}_r
\]

\[
H_r = C_r V (sI - \Lambda)^{-1} V^{-1} \tilde{B}_r.
\]

(4.25)

Since \( sI - A \) is diagonal, \( (sI - A)^{-1} \) is diagonal with the diagonal entries being \( \frac{1}{s - \lambda_i} \), where \( \lambda_i \) is the \( i \)th eigenvalue of \( \tilde{A}_r \), and thus the generalized eigenvalue of \( A \) and \( E \). This leads to definitions for \( l_i \) and \( r_i \) as the \( i \)th columns of \( C_r V \) and \( V^{-1} E^{-1} B_r \) respectively. This expansion plays a key part in choosing the interpolation points and direction vectors as we find that if \( H_r(s) \) is the optimal reduced order approximation of \( H(s) \), then \( H_r(s) \) satisfies
the conditions of 4.21 with the interpolation points defined as \( \{ \sigma_i \}_{i=1}^r = \{-\lambda_i\}_{i=1}^r \) and the left and right directional vectors defined as the residue \( \mathbf{l} \) and \( \mathbf{r} \) vectors from 4.22.

5 IRKA

Because the optimal interpolation points and directional vectors depend on the reduced model, we require an iterative process that converges to the optimal reduced model. That iterative procedure to given as the Iterative Rational Krylov Algorithm (IRKA): Given initial sets of interpolation points \( \{ \sigma_i \}_{i=1}^r \in \mathbb{C} \), right directional vectors \( \{ \mathbf{r}_i \}_{i=1}^r \in \mathbb{C}^m \), and left directional vectors \( \{ \mathbf{l}_i \}_{i=1}^r \in \mathbb{C}^p \), the \( \mathbf{V} \) and \( \mathbf{W} \) basis are constructed as

\[
\mathbf{V} = \begin{bmatrix} (\sigma_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \mathbf{r}_1 & \cdots & (\sigma_r \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \mathbf{r}_r \end{bmatrix}
\]

\[
\mathbf{W} = \begin{bmatrix} (\sigma_1 \mathbf{E}^T - \mathbf{A}^T)^{-1} \mathbf{C}^T \mathbf{l}_1 & \cdots & (\sigma_r \mathbf{E}^T - \mathbf{A}^T)^{-1} \mathbf{C}^T \mathbf{l}_r \end{bmatrix}
\]

Then using the definitions used in 3.19, we form the reduced transfer function \( \mathbf{H}_r \) as in 2.8. Then the pole-residue of \( \mathbf{H}_r(s) \) is computed as

\[
\mathbf{H}_r(s) = \sum_{i=1}^r \frac{\hat{\mathbf{l}}_i \hat{\mathbf{r}}^T_i}{s - \lambda_i}
\]

and new sets of interpolation points and directional vectors are defined as

\[
\{ \sigma_i \}_{i=1}^r = \{-\lambda_i\}_{i=1}^r
\]

\[
\{ \mathbf{r}_i \}_{i=1}^r = \{ \hat{\mathbf{r}}_i \}_{i=1}^r
\]

\[
\{ \mathbf{l}_i \}_{i=1}^r = \{ \hat{\mathbf{l}}_i \}_{i=1}^r
\]

which in turn constructs new \( \mathbf{V} \) and \( \mathbf{W} \) bases. This process can be repeated until convergence is achieved. Measuring the state of convergence can be done by looking at the difference between newly constructed interpolation points denoted \( \sigma^k \) and the previously used interpolation points \( \sigma^{k-1} \) and scaling it with the norm of the constructed interpolation points, then comparing it with a set threshold parameter \( D \). Thus convergence is said to be reached once

\[
\frac{\| \sigma^k - \sigma^{k-1} \|}{\| \sigma^k \|} < D
\]

6 Implementation

The IRKA process has been implemented successfully in both SISO and MIMO cases. First we look at a SISO system with its full order transfer function being

\[
\mathbf{H}(s) = \frac{2s^6 + 11.5s^5 + 57.75s^4 + 178.6s^3 + 345.5s^2 + 323.6s + 94.5}{s^7 + 10s^6 + 46s^5 + 130s^4 + 239s^3 + 280s^2 + 194s + 69}
\]
which means its dimension is $n = 7$. Running IRKA for reduced orders of $r = 4$ and $r = 6$ successful yielded reduced transfer functions $H_4$ and $H_6$ which approximate $H$ to a high level of accuracy. The plots of $H_4$ and $H_6$ are shown below, including $H$ for comparison.

As one can see, both reduced models approximate the original model well, but $H_6$ follows the original more closely, specially at local minimal point.

Now we will look at several MIMO systems. First we look at a model of the heat of a steel bar, with a dimension of $n = 197$ and two outputs. The three following plots show the reduced transfer function with orders $r = 2$, $r = 5$ and $r = 8$. 

![Graph showing SISO Model with reduced dimensions r=4 and r=6](image1)

![Graph showing Heat Model](image2)
As with the previous SISO model, the accuracy of the approximation increases as the reduced order increases. Note that in all three plots, one output is approximated more accurately.

Now we examine a model of dimension of $n = 100$, also with two outputs, with describes the behavior of a system of mass spring dampers. Reduced models of order $r = 3$, $r = 7$ and $r = 15$ are shown, and as with the previous two models, one can see the accuracy of the model increases as the reduced order increases.
As with the heat model, one of the outputs is better approximated in each of the reduced models.

7 Future Research

In the future, I would like to try to apply what I have learned regarding reduction of models of the form 1.1 to the parametric case, where the constant matrices are replaced by parameter dependent matrices, and the state variable and output may depend on both \( t \) and that parameter. This case can be written as

\[
E(p) \dot{x}(t, p) = A(p)x(t, p) + B(p)u(t) \\
y(t, p) = C(p)x(t, p)
\] (7.30)

where \( p = [p_1, \ldots p_v] \) is a length \( v \) vector of parameters. This problem arises in many areas of science and due to the parameter dependency of the matrices, finding approximations that hold for all cases can be difficult.