Eigenfunctions of Composition and Weighted Composition Operator on \(\alpha\)-Bloch Spaces

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\( \mathbb{D} \) is the unit disk of \( \mathbb{C} \).

\( H(\mathbb{D}) \) is the set of all holomorphic functions on \( \mathbb{D} \).

\( \varphi \) is a holomorphic self map of \( \mathbb{D} \).

The composition operator with symbol \( \varphi \) is defined as

\[
C_\varphi(f)(z) = (f \circ \varphi)(z)
\]

Schröder's functional equation

\[
C_\varphi(f)(z) = (f \circ \varphi)(z) = \lambda f(z) \quad (1)
\]

where \( \lambda \neq 0 \) is complex constant.

A natural question is "What are \( \lambda \) and \( f \) that satisfy (1)?"
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A natural question is “What are \( \lambda \) and \( f \) that satisfy (1)?”
If $\phi$ is a holomorphic self map of $\mathbb{D}$ such that $\phi(0) = 0$, and $0 < |\phi'(0)| < 1$, then
Königs’s Theorem (1884)

If $φ$ is a holomorphic self map of $\mathbb{D}$ such that $φ(0) = 0$, and $0 < |φ'(0)| < 1$, then

1. The sequence of holomorphic functions

$$σ_k(z) := \frac{φ_k(z)}{φ'(0)^k}, \quad \text{where } φ_k = φ \circ φ \circ ... \circ φ$$

converges uniformly on compact subsets of $\mathbb{D}$ to a non-constant holomorphic function $σ$ and $σ$ satisfies (1) with $λ = φ'(0)$. 

2. If $f$ is a non-constant holomorphic function that satisfies (1), then there exists a positive integer $n$ such that $λ = φ'(0)^n$ and $f$ is a constant multiple of $σ^n$. 

Note: $σ$ is known as König's function.
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ii) If $f$ is a non-constant holomorphic function that satisfies (1), then there exists a positive integer $n$ such that $\lambda = \phi'(0)^n$ and $f$ is a constant multiple of $\sigma^n$. 
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2. If \( f \) is a non-constant holomorphic function that satisfies (1), then there exists a positive integer \( n \) such that \( \lambda = \phi'(0)^n \) and \( f \) is a constant multiple of \( \sigma^n \).

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Assume $\phi(0) = 0$, $0 < |\phi'(0)| < 1$ and $\sigma$ is Königs function. Let $C_\phi$ be bounded on $\alpha$-Bloch space $B_\alpha$, $\alpha \in \mathbb{R}_+$. What are the conditions on $\phi$ that ensure all these eigenfunctions $\sigma^n \in B_\alpha$.

Similar Problem for weighted composition operators.
For \( \alpha \in \mathbb{R}_+ \), \( \alpha \)-Bloch is defined as
\[
\mathcal{B}_\alpha = \{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty \}
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$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)|$$
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- $|f(z)| \leq c_f \log \frac{2}{1-|z|}$, $\forall f \in \mathcal{B}_1 = \mathcal{B}$ (say) and $c_f$ constant
- $\| f \|_{\mathcal{B}_\alpha} \approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha-1} |f(z)|$ for $\alpha > 1$
Known result:
if $C_\phi$ is compact operator on $B_\alpha$ for $\alpha > 0$ then then $\sigma^n \in B_\alpha$ for all $n$. 
Definition

Hyperbolic $\alpha$-derivative of $\phi$ is defined by

$$
\phi^{(h_{\alpha})}(z) = \frac{(1 - |z|^2)^{\alpha} \phi'(z)}{(1 - |\phi(z)|^2)^{\alpha}}.
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**Definition**

Let us say $\phi$ satisfies condition (A) if there exits a non negative integer $m$ such that
\[
|\phi^{(h_\alpha)}(\phi_m(z))| = \frac{(1 - |\phi_m(z)|^2)^\alpha \phi'(\phi_m(z))}{(1 - |\phi_{m+1}(z)|^2)^\alpha} \leq |\phi'(0)|, \quad \text{for all } z \in \mathbb{D}
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(A)

Remark

If $m = 0$, $|\phi^{(h_\alpha)}(z)| \leq |\phi'(0)|$, for all $z \in \mathbb{D}$
For \( \mathcal{B}_\alpha, \ 0 < \alpha < 1 \)

**Theorem 1**

Let \( 0 < \alpha < 1 \), and \( C_\phi \) is bounded on \( \mathcal{B}_\alpha \).

- If there exists \( k \in \mathbb{N} \) such that \( \| \phi_k \|_\infty < 1 \) then \( \sigma^n \in \mathcal{B}_\alpha \) for all \( n \in \mathbb{N} \).

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Let $0 < \alpha < 1$, and $C_\phi$ is bounded on $\mathcal{B}_\alpha$.

- if there exists $k \in \mathbb{N}$ such that $\|\phi_k\|_\infty < 1$ then $\sigma^n \in \mathcal{B}_\alpha$ for all $n \in \mathbb{N}$.
- If $\phi$ satisfies condition (A) then $\sigma^n \in \mathcal{B}_\alpha$ for all $n \in \mathbb{N}$.

**Example**

$$\phi(z) = \frac{iz(z + 1)}{2}$$

Here, $\phi(0) = 0$, $\phi'(0) = \frac{i}{2}$ and $\|\phi_2\|_\infty < 1$. 
Consider a map \( \gamma(z) = \frac{1 + z}{1 - z} \) that maps unit disk to left half plane univalently.
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Consider a map $\gamma(z) = \frac{1 + z}{1 - z}$ that maps unit disk to left half plane univalently and suppose $\phi_t(z) = \frac{\gamma(z)^t - 1}{\gamma(z)^t + 1}$ where $t \in (0, 1)$ that maps unit disk to unit disk.
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\[
|\phi_t^{(h)}(z)| = \frac{(1 - |z|^2) |\phi_t'(z)|}{(1 - |\phi_t(z)|^2)} \leq |\phi_t'(0)|
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Therefore \( \phi_t \) satisfies condition (A) for \( \alpha = 1 \) and \( m = 0 \).
Consider a map $\gamma(z) = \frac{1 + z}{1 - z}$ that maps unit disk to left half plane univalently and suppose $\phi_t(z) = \frac{\gamma(z)^t - 1}{\gamma(z)^t + 1}$ where $t \in (0, 1)$ that maps unit disk to unit disk. Here, $\phi_t(0) = 0$, $\phi'_t(0) = t$

$$|\phi_t^{(h)}(z)| = \frac{(1 - |z|^2)|\phi'_t(z)|}{(1 - |\phi_t(z)|^2)} \leq |\phi'_t(0)|$$

Therefore $\phi_t$ satisfies condition (A) for $\alpha = 1$ and $m = 0$.

Note that Kőnigs’ function of $C_{\phi_t}$ is $\sigma(z) = \log \frac{1 + z}{1 - z} \in \mathcal{B}$
For $B_\alpha$, $\alpha > 1$

**Theorem 2**

If $\phi$ satisfies condition (A) for $\alpha = 1$ then $\sigma \in B$. 

**Definition 1**

Bloch number of a function $\sigma$ is $b = \inf \alpha \{ \alpha : \sigma \in B_\alpha \}$.

**Theorem 3**

Let $\alpha > 1$. $\sigma^*_n \in B_\alpha$ for all $n \in \mathbb{N}$ if only if Bloch number $b$ of $\sigma$ is at most 1.

**Corollary 1**

If $\phi$ satisfies condition (A) for $\alpha = 1$ then $\sigma^*_n \in B_\beta$ for all $\beta > 1$ and for all $n \in \mathbb{N}$. 

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**Corollary 1**
If $\phi$ satisfies condition (A) for $\alpha = 1$ then $\sigma^n \in \mathcal{B}_\beta$ for all $\beta > 1$ and for all $n \in \mathbb{N}$.
For $\mathcal{B}$

**Theorem 4**

Assume

\[
\frac{(1 - |z|^2) |\phi'(z)|}{(1 - |\phi(z)|^2)} \frac{\log \frac{2}{1 - |z|}}{\log \frac{2}{1 - |\phi(z)|}} \leq |\phi'(0)|
\]

then \( \{\sigma^n\}_{n=0}^{\infty} \subset \mathcal{B} \)
Weighted composition operator

**Definition**

Let $u$ a holomorphic function in $\mathbb{D}$, then the operator defined by the relation

$$(uC_\phi)(f)(z) = u(z)f(\phi(z))$$

for $f \in H(\mathbb{D})$ is called weighted composition operator.
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**Schröder Equation**

The Schröder equation for wt. composition operator is

$$u(z)f(\phi(z)) = \lambda f(z)$$

Where $f$ is holomorphic function and $\lambda \neq 0$. 
Theorem 5

(T. Hosokawa and Q. D. Nguyen, 2010)

If \( u(0) \neq 0, \ \phi(0) = 0, \ 0 < |\phi'(0)| < 1 \), then

- **The Sequence of holomorphic function**

\[
v_k(z) = \frac{u(z)u(\phi(z))\ldots u(\phi_{k-1}(z))}{u(0)^k}
\]

converges uniformly on compact subset of \( \mathbb{D} \) to a holomorphic function \( v \) and \( v \) satisfies (3) with \( \lambda = u(0) \).
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- If \( f \) is a holomorphic function that satisfies (3), then there exists a positive integer \( n \) such that

\[
\lambda = u(0)\phi'(0)^n
\]

and \( f \) is a constant multiple of \( v\sigma^n \), where \( \sigma \) is Königs function.
Theorem 6

(T. Hosokawa and Q. D. Nguyen, 2010)

Assume that \( u \in C \phi \) is bounded on \( B \). For \( 0 < r < 1 \), set

\[
a_r := \sup_{|z| = r} \left\{ |u'(z) \phi(z)| + |u(z) \phi'(z)| \right\}.
\]

Further assume that

\[ i \lim_{r \to 1} \log(1 - r) \log M_r(\phi) = \infty. \]

\[ ii \log |a_r| < \epsilon \log(1 - r) \log M_r(\phi), \text{ where } \epsilon > 0 \text{ is a constant satisfying } \epsilon \log \| \phi \|_\infty > -1. \]

Then, \( v \sigma^n \in B \) for all \( n \in \mathbb{N} \).

Remark

It can be shown that these conditions are strong enough to imply \( u \in C \phi \) is compact on \( B \).
Theorem 7

Assume $0 < \alpha < 1$ and $u \in C_\phi$ bounded on $B_\alpha$. If $\|u'\|_\infty < \infty$ and there exists $K \in \mathbb{N}$ such that $\|\phi_k\|_\infty < 1$ then $v \sigma^n \in B_\alpha$ for all $n$.
Theorem 7
Assume $0 < \alpha < 1$ and $u \in C^{\phi}$ bounded on $B_{\alpha}$. If $\|u'\|_{\infty} < \infty$ and there exists $K \in \mathbb{N}$ such that $\|\phi_k\|_{\infty} < 1$ then $v\sigma^n \in B_{\alpha}$ for all $n$.

Theorem 8
$(H.,N.)$ If $\|u\|_{\infty} < \infty$ and $\|\phi_k\|_{\infty} < 1$ for some $k \in \mathbb{N}$ then $v\sigma^n \in B$ for all $n$. 
Theorem 9

For $\alpha > 1$, suppose $|u(z)| \leq \frac{(1-|z|^2)^{\alpha-1}}{(1-|\phi(z)|^2)^{\alpha-1}} \leq |u(0)|$
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Theorem 9

For $\alpha > 1$, suppose $|u(z)| \frac{(1-|z|^2)^{\alpha-1}}{(1-|\phi(z)|^2)^{\alpha-1}} \leq |u(0)|$

i. If $\|\phi_k\|_{\infty} < 1$ for some $k \in \mathbb{N}$ then $v \sigma^n \in \mathcal{B}_\alpha$.

ii. If $|\phi^{(h)}(z)| \leq |\phi'(0)|$ then $v \sigma^n \in \mathcal{B}_p$, for any $p > \alpha$. 
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For $\alpha > 1$, suppose $|u(z)| \frac{(1-|z|^2)^{\alpha-1}}{(1-|\phi(z)|^2)^{\alpha-1}} \leq |u(0)|$

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Theorem 10

For $\alpha > 1$, suppose

i. $|u(z)| \frac{(1-|z|^2)^{\alpha-1}}{(1-|\phi(z)|^2)^{\alpha-1}} \frac{\log \frac{2}{1-|z|}}{\log \frac{2}{1-|\phi(z)|}} \leq |u(0)|$
**Theorem 9**

For $\alpha > 1$, suppose $|u(z)| \frac{(1-|z|^2)^{\alpha-1}}{(1-|\phi(z)|^2)^{\alpha-1}} \leq |u(0)|$

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2. $|\phi^{(h)}(z)| \frac{\log \frac{2}{1-|z|}}{\log \frac{2}{1-|\phi(z)|}} \leq |\phi'(0)|$

then $v\sigma^n \in B_\alpha$ for all $n$. 
THANK YOU