Sampling and Interpolation on Some Nilpotent Lie Groups

SEAM 2013

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March 2013
A Prominent Example (Whittaker, Shannon, Kotel’nikov)

Let $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. The **Paley Wiener** space

$$\text{PW} = \left\{ f \in L^2(\mathbb{R}) : \text{support } \hat{f} \subset \left[ -\frac{1}{2}, \frac{1}{2} \right] \right\}$$

1. The map $f \mapsto (f(k))_{k \in \mathbb{Z}}$ is an isometry.
2. Any function in $\text{PW}$ is completely determined by its values on $\mathbb{Z}$. The series

$$f(t) = \sum_{k \in \mathbb{Z}} f(k) \text{sinc}(x - k)$$

converges unif. in $L^2(\mathbb{R})$.

3. $\{\text{sinc}(x - k) : k \in \mathbb{Z}\}$ is an ONB.

4. $\text{PW}$ is a **sampling space** with **interpolation property**.
A Starting Point

Let $G$ be a locally compact group, $\Gamma \subset G$. Let $H \subset L^2(G)$ be a left-invariant closed subspace of $L^2(G)$ consisting of continuous functions. $H$ is a sampling space with respect to $\Gamma$ if

1. $f \mapsto (f(\gamma))_{\gamma \in \Gamma}$ is an isometry: $\sum_{\gamma \in \Gamma} |f(\gamma)|^2 = \|f\|^2$.

2. There exists $S \in H$ such that for all $f \in H$,

$$f(x) = \sum_{\gamma \in \Gamma} f(\gamma) S(\gamma^{-1}x).$$

and $S$ is called a sinc-type function.

3. A sampling space has the interpolation property if $f \mapsto (f(\gamma))_{\gamma \in \Gamma}$ is unitary.
The Heisenberg group

B. Currey and A. Mayeli proved in 2012 that the Heisenberg group admits sampling spaces with interpolation property with respect to $\Gamma$

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} z \\ y \\ x \end{bmatrix} \in \mathbb{R}^3 \right\}$$

$$\Gamma = \left\{ \begin{bmatrix} 1 & m & k \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} k \\ l \\ m \end{bmatrix} \in \mathbb{Z}^3 \right\}$$

The interpolation property was a surprising fact !!!
Let $N$ be a locally compact group. We consider $L^2(N)$ with left regular representation $L$.

$$L(y)f(x) = f(y^{-1}x)$$

1. How do we generalize the concept of bandlimitation?
2. How do we pick a subset $\Gamma \subset N$ such that $f \mapsto (f(\gamma))_{\gamma \in \Gamma}$ is an isometry?
3. What is the structure of $\Gamma$?
4. Do sampling subspaces of $L^2(N)$ with respect to $\Gamma$ exist? If yes do they have the interpolation property?
5. How do we construct sampling subspaces of $L^2(N)$?
Sampling on Non-abelian groups

2. B. Currey, A. Mayeli, A Density Condition for Interpolation on the Heisenberg Group, Rocky Mountain Journal of Mathematics
7. V. Oussa, Sampling and Interpolation Conditions on Some Two-Step Nilpotent Lie Groups, preprint
Content of the talk

1. Consider a class of groups of non-commutative unipotent matrices
2. Develop the concept of bandlimitation
3. Show existence of sampling spaces
4. Give conditions for interpolation
Let $N$ be a non-commutative simply connected, connected two step nilpotent Lie group with Lie algebra $\mathfrak{n}$

1. $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{b} \oplus \mathfrak{a}$, $\mathfrak{z}$ is the center of $\mathfrak{n}$.

2. $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{z}$ where $\mathfrak{b} = \bigoplus_{k=1}^{d} \mathbb{R} \mathfrak{Y}_k$, $\mathfrak{a} = \bigoplus_{k=1}^{d} \mathbb{R} \mathfrak{X}_k$.

3. $\mathfrak{a}, \mathfrak{b}$ are commutative algebras.

4. $\det \left( \begin{bmatrix} [X_1, Y_1] & \cdots & [X_1, Y_d] \\ \vdots & \ddots & \vdots \\ [X_d, Y_1] & \cdots & [X_d, Y_d] \end{bmatrix} \right)$ is a **non-vanishing polynomial** over $[\mathfrak{n}, \mathfrak{n}]$.
Some Facts

Theorem

If $N$ satisfies the previous conditions then $N \cong \mathbb{R}^{n-d} \rtimes \mathbb{R}^d$ and there is finite dimensional faithful matrix representation of $N$ in $GL(n+1, \mathbb{R})$. 
Example 1

The Heisenberg Lie group with Lie algebra spanned by \( \{Z, Y, X\} \) with Lie brackets \([X, Y] = Z\)

\[
N \cong \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} z \\ y \\ x \end{bmatrix} \in \mathbb{R}^3 \right\}
\]
Example 2

Let

\[ N = \exp(\mathbb{R}Z_1 \oplus \mathbb{R}Z_2) \exp(\mathbb{R}Y_1 \oplus \mathbb{R}Y_2) \exp(\mathbb{R}X_1 \oplus \mathbb{R}X_2) \]

with Lie brackets

\[
[X_1, Y_1] = Z_1, [X_2, Y_1] = -Z_2, [X_1, Y_2] = Z_2, [X_2, Y_2] = Z_1.
\]

Here is a matrix representation of \( N \).

\[
N \cong \left\{ \begin{bmatrix}
1 & 0 & x_1 & x_2 & -y_1 & -y_2 & z_1 \\
0 & 1 & -x_2 & x_1 & -y_2 & y_1 & z_2 \\
0 & 0 & 1 & 0 & 0 & 0 & y_1 \\
0 & 0 & 0 & 1 & 0 & 0 & y_2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} : \begin{bmatrix}
z_1 \\
z_2 \\
y_1 \\
y_2 \\
x_1 \\
x_2
\end{bmatrix} \in \mathbb{R}^6 \right\}.
\]
Frames

Given a countable sequence $\{f_i\}_{i \in I}$ of functions in a Hilbert space $\mathcal{H}$, we say $\{f_i\}_{i \in I}$ forms a frame iff there exist strictly positive real numbers $A, B$ s.t.

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2$$

1. If $A = B$, $\{f_i\}_{i \in I}$ is a tight frame.
2. If $A = B = 1$, $\{f_i\}_{i \in I}$ is a Parseval frame.
3. If $\{f_i\}_{i \in I}$ is a Parseval frame such that for all $i \in I$, $\|f_i\| = 1$ then $\{f_i\}_{i \in I}$ is an orthonormal basis for $\mathcal{H}$. 
Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a full rank separable lattice in $\mathbb{R}^{2d}$ and $g \in L^2 (\mathbb{R}^d)$. The system

$$G \left( g, A\mathbb{Z}^d \times B\mathbb{Z}^d \right) = \left\{ e^{2\pi i \langle k, x \rangle} g (x - n) : k \in B\mathbb{Z}^d, n \in A\mathbb{Z}^d \right\}$$

is called a **Gabor system**.
Density Condition

Given a separable full rank lattice \( \Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d \) in \( \mathbb{R}^{2d} \). The following facts are equivalent

1. There exists \( \phi \in L^2(\mathbb{R}^d) \) such that \( \mathcal{G}(\phi, \Lambda) \) is a Parseval frame in \( L^2(\mathbb{R}^d) \).
2. \( \text{vol}(\Lambda) = |\det A \det B| \leq 1. \)
Harmonic Analysis on $\mathbb{N}$

The **Fourier transform** is defined on $L^2(N) \cap L^1(N)$ by

$$\mathcal{P}(f)(\lambda) = \int_{\Sigma} \pi_{\lambda}(n)f(n)\,dn,$$

where for $B(\lambda) = (\lambda [X_i, Y_j])_{1 \leq i,j \leq d}$

1. $\Sigma$ is a Zariski open subset of the linear dual of the $\mathfrak{z}$

   $$\Sigma = \{ \lambda \in \mathfrak{z}^* : \det(B(\lambda)) \neq 0 \}$$

2. $\hat{\mathfrak{N}} = \{ \pi_{\lambda} : \lambda \in \Sigma \}$ where

   $$\pi_{\lambda}(\exp a \exp b)f = \left\{ e^{-2\pi i \langle B(\lambda)y, t \rangle}f(t-x) : y \in \mathbb{R}^d, x \in \mathbb{R}^d \right\}$$

   $$\pi_{\lambda}(\exp \mathfrak{z})f = \left\{ e^{2\pi i \lambda(z)}f : z \in \mathbb{R}^{n-2d} \right\} \quad \text{and} \quad f \in L^2\left(\mathbb{R}^d\right)$$
Harmonic Analysis on $N$

1. $\mathcal{P} (L^2(N)) = \int_{\Sigma}^\oplus L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \, d\mu(\lambda)$

2. $\mathcal{P} \circ L \circ \mathcal{P}^{-1} = \int_{\Sigma}^\oplus \pi_\lambda \otimes 1_{L^2(\mathbb{R}^d)} \, d\mu(\lambda)$

3. The Plancherel measure: $d\mu(\lambda) = |\det(B(\lambda))| \, d\lambda$ where

$$B(\lambda) = \begin{bmatrix} \lambda [X_1, Y_1] & \cdots & \lambda [X_1, Y_d] \\ \vdots & \ddots & \vdots \\ \lambda [X_d, Y_1] & \cdots & \lambda [X_d, Y_d] \end{bmatrix}$$

4. Plancherel Theorem $\|f\|^2_{L^2(N)} = \int_{\Sigma} \|\mathcal{P}(f)(\lambda)\|^2_{HS} \, d\mu(\lambda)$. 
Example

\[ N = \exp(\mathbb{R}Z_1 \oplus \mathbb{R}Z_2) \exp(\mathbb{R}Y_1 \oplus \mathbb{R}Y_2) \exp(\mathbb{R}X_1 \oplus \mathbb{R}X_2) \] with

\[ [X_1, Y_1] = Z_1, [X_1, Y_2] = Z_2, \]
\[ [X_2, Y_1] = Z_2, [X_2, Y_2] = Z_1. \]

Then

1. \( \Sigma = \{(\lambda_1, \lambda_2, 0, 0, 0, 0) \in n^* : \lambda_1^2 + \lambda_2^2 \neq 0\} \)
2. The Plancherel measure is \((\lambda_1^2 + \lambda_2^2) d\lambda_1 d\lambda_2\)
Theorem

(Oussa, 2012) For a fixed \( \lambda \in \Sigma \), let \( \mathcal{L}_a \) be a lattice in \( a \cong \mathbb{R}^d \) and let \( \mathcal{L}_b \) be a lattice in \( b \cong \mathbb{R}^d \).

\[
\pi_\lambda (\exp \mathcal{L}_a \exp \mathcal{L}_b) f
\]

is a Gabor system.
Concept of Bandlimitation

1. We say a Hilbert space $H$ is a **multiplicity-free** left-invariant closed subspace of $L^2(N)$ if and only if

$$\hat{H} = \int_{\Sigma}^\oplus L^2(\mathbb{R}^d) \otimes f_\lambda \ d\mu(\lambda)$$

where $\{f_\lambda : \lambda \in \Sigma, \|f_\lambda\| = 1\}$

2. We say a function $\phi \in L^2(N)$ is **bandlimited** if its Plancherel transform is supported on a bounded measurable subset of $\Sigma$. 
Lattice in $\mathbb{N}$

**Lemma**

If $\mathfrak{n}$ has a **rational structure** and

$$\Gamma = \prod_{k=1}^{n-2d} \exp(\mathbb{Z}Z_k) \prod_{k=1}^{d} \exp(\mathbb{Z}Y_k) \prod_{k=1}^{d} \exp(\mathbb{Z}X_k)$$

then there is a choice of Lie algebra for which the group generated by $\Gamma$ is a lattice subgroup of $\mathbb{N}$.
Theorem

(Oussa) Define $E = \{ \lambda \in \Sigma : |\det B(\lambda)| \leq 1 \}$ and

$$\hat{H}_{K \cap E} = \int_{K \cap E} \oplus L^2(\mathbb{R}^d) \otimes f_\lambda |\det B(\lambda)| d\lambda.$$ 

Assume that $K$ is compact s.t $\left\{ e^{2\pi i \langle k, \lambda \rangle} \chi_K(\lambda) \right\}$ is a Parseval frame for $L^2(K, d\lambda)$. Then there exists $\phi \in H_{K \cap E}$ such that

$$\hat{\phi}(\lambda) = \frac{1}{\sqrt{|\det B(\lambda)|}} (g_\lambda \otimes f_\lambda)$$

and the system $G(g_\lambda, \mathbb{Z}^d \times B(\lambda) \mathbb{Z}^d)$ is a Gabor Parseval frame in $L^2(\mathbb{R}^d)$ for all $\lambda \in K \cap E$, and $L(\Gamma) \phi$ is a Parseval frame in $H_{K \cap E}$. 

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Existence of Sampling Spaces with Sinc-type Functions.

Define $\mathcal{W}_\phi : \mathbb{H}_{K \cap E} \rightarrow L^2 (N)$ where

$$\mathcal{W}_\phi f (n) = \langle f, L(n) \phi \rangle = f * \phi^* (n) \text{ and } \phi^* (x) = \overline{\phi (x^{-1})}$$

Recall

$$\Gamma = \prod_{k=1}^{n-2d} \exp (ZZ_k) \prod_{k=1}^{d} \exp (ZY_k) \prod_{k=1}^{d} \exp (ZX_k).$$

**Theorem**

*(Oussa, 2012)* The Hilbert space $\mathcal{W}_\phi (H_{K \cap E})$ is a $\Gamma$-sampling space with sinc-type function $\phi * \phi^*$
An Example

1. $N = \prod_{k=1}^{2} \exp(\mathbb{R}Z_k) \prod_{k=1}^{2} \exp(\mathbb{R}Y_k) \prod_{k=1}^{2} \exp(\mathbb{R}X_k)$

2. $\Gamma = \prod_{k=1}^{2} \exp(\mathbb{Z}Z_k) \prod_{k=1}^{2} \exp(\mathbb{Z}Y_k) \prod_{k=1}^{2} \exp(\mathbb{Z}X_k)$

3. $[X_1, Y_1] = Z_1, [X_2, Y_2] = Z_1, [X_2, Y_1] = Z_2, [X_1, Y_2] = Z_2.$
A matrix realization of the group

\[
N = \begin{bmatrix}
1 & 0 & x_1 & x_2 & -y_1 & -y_2 & z_1 \\
0 & 1 & x_2 & x_1 & -y_2 & -y_1 & z_2 \\
0 & 0 & 1 & 0 & 0 & 0 & y_1 \\
0 & 0 & 0 & 1 & 0 & 0 & y_2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
:\begin{bmatrix}
z_1 \\
z_2 \\
y_1 \\
y_2 \\
x_1 \\
x_2
\end{bmatrix}
\]  

\( E = \{ (\lambda_1, \lambda_2, 0, 0, 0, 0) : |\lambda_1^2 - \lambda_2^2| \leq 1, \text{ and } |\lambda_1^2 - \lambda_2^2| \neq 0 \} \)
There is a **restriction** on how to pick $K$. The choice of $K$ depends on the structure constants of $\mathfrak{n}$.

Define $\hat{\phi}(\lambda) = \frac{1}{\sqrt{|\lambda_1^2 - \lambda_2^2|}} g_{\lambda} \otimes f_{\lambda}$

$G \left( g_{\lambda}, \mathbb{Z}^2 \times \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \mathbb{Z}^2 \right)$ is a Parseval Gabor frame

$W_{\phi}(H_{E \cap K})$ is a sampling space with Sinc-type function $\phi \ast \phi^*$
An Example (Part I)

\[ N = \prod_{k=1}^{2} \exp (\mathbb{I}Z_k) \prod_{k=1}^{3} \exp (\mathbb{I}Y_k) \prod_{k=1}^{3} \exp (\mathbb{I}X_k) \]

\[ \Gamma = \prod_{k=1}^{2} \exp (\mathbb{Z}Z_k) \prod_{k=1}^{3} \exp (\mathbb{Z}Y_k) \prod_{k=1}^{3} \exp (\mathbb{Z}X_k) \]

\[
\begin{align*}
[X_1, Y_1] &= Z_1, [X_2, Y_1] = Z_2, [X_3, Y_1] = Z_2 \\
[X_1, Y_2] &= Z_2, [X_2, Y_2] = Z_2, [X_3, Y_2] = Z_1 \\
[X_1, Y_3] &= Z_2, [X_2, Y_3] = Z_1, [X_3, Y_3] = Z_2.
\end{align*}
\]

\[ E = \left\{ (\lambda_1, \lambda_2, 0, 0, 0, 0) : \begin{aligned}
&| -\lambda_1^3 + 3\lambda_1\lambda_2^2 - 2\lambda_2^3 | \leq 1, \\
&| -\lambda_1^3 + 3\lambda_1\lambda_2^2 - 2\lambda_2^3 | \neq 0
\end{aligned} \right\} \]
Acceptable Bandlimitation

This Set K will not work

This Set will work
A Necessary Condition for Interpolation.

Theorem

(Oussa, 2012) Let $H_I$ be a bandlimited multiplicity-free space such that

$$\hat{H}_I = \int_{I}^{\oplus} L^2 \left( \mathbb{R}^d \right) \otimes f_{\lambda} |\det B(\lambda)| \, d\lambda.$$ 

If $H_I$ is a sampling space with the interpolation property then

1. there exists $\phi \in H_I$ such that $L(\Gamma) \phi$ is a Parseval frame
2. $\int_{I} |\det B(\lambda)| \, d\lambda = 1.$