Compactness of operators on generalized Fock spaces

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Outline of this talk

- Section 1: Introduction
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- Section 2: Main results
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- Section 2: Main results
- Section 3: Open problems
Generalized Fock spaces

For \( z, w \in \mathbb{C}^n \), let \( z \cdot \overline{w} \) be the usual anti-linear dot product

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z \cdot \overline{w} := z_1 \overline{w_1} + \cdots + z_n \overline{w_n}
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and let \( |z|^2 = z \cdot \overline{z} \).
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Let \( \phi \in C^2(\mathbb{C}^n) \) satisfy \( 0 < m < \Delta \phi < M \).
Generalized Fock spaces

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  and let $|z|^2 = z \cdot \overline{z}$.
- Let $\phi \in C^2(\mathbb{C}^n)$ satisfy $0 < m < \Delta \phi < M$.
- If $0 < p < \infty$ then let $L^p_\phi := \{ f : f(\cdot) e^{-\phi(\cdot)} \in L^p(\mathbb{C}^n, dv) \}$. 
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If $0 < p < \infty$ then let $L^p_\phi := \{ f : f(\cdot)e^{-\phi(\cdot)} \in L^p(\mathbb{C}^n, dv) \}$.

Let $F^p_\phi := \{ f \text{ entire on } \mathbb{C}^n : f \in L^p_\phi \}$. 
**Generalized Fock spaces**

- Example: if $\phi(z) = \frac{|z|^2}{8t}$ for $t > 0$, then $\Delta(|z|^2/8t) = n/2t$ and $F^p_t := F^p_\phi$ is the classical Fock space.
Generalized Fock spaces

- Example: if \( \phi(z) = \frac{|z|^2}{8t} \) for \( t > 0 \), then \( \Delta \left( \frac{|z|^2}{8t} \right) = \frac{n}{2t} \) and \( F_t^p := F_\phi^p \) is the classical Fock space.

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Nontrivial example (Fock-Sobolev spaces): If $m \in \mathbb{N}$ then let $F^{p}_{t,m}$ be the Banach space of entire functions where

$$\sum_{|\alpha| \leq m} \left\| (\partial^\alpha f)(\cdot) e^{-\frac{|\cdot|}{8t}} \right\|_{L^p(\mathbb{C}^n, dv)}$$
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then $F^p_{t,m}$ can be written as a generalized Fock space (J.I., preprint).
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- Note: $F^2_\phi$ is a reproducing kernel Hilbert space under the canonical Hilbert space inner product

$$\langle f, g \rangle_{F^2_\phi} = \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-2\phi(z)} \, dv(z).$$
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- Let $K(z, w)$ be the reproducing kernel of $F^2_\phi$ and let $k_w(z) := K(z, w)/\sqrt{K(w, w)}$ be the normalized reproducing kernel of $F^2_\phi$. 
Toeplitz operators

The orthogonal projection $P : L^2_{\phi} \to F^2_{\phi}$ is given by

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$$T_\mu f(z) = \int_{\mathbb{C}^n} K(z, w)f(w)e^{-2\phi(w)} \, d\mu(w).$$
Berezin transform

- Note: $\sup_{w \in \mathbb{C}^n} \| k_w \|_{F^p_\phi} < \infty$ for any $0 < p < \infty$ 
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- Note: \((F^p_\phi)^* = F^q_\phi\) under the natural pairing induced by \( \langle \cdot, \cdot \rangle_{F^2_\phi} \) if \( 1 < p < \infty \) (Schuster/Varolin, '12).
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- Note: $\lim_{|w| \to \infty} k_w = 0$ weakly as $|w| \to \infty$ (Schuster/Varolin, ’12).
- Berezin transform: if $A$ is bounded on $F^p_\phi$ for $1 < p < \infty$ then let $B(A)(w) := \langle Ak_w, k_w \rangle_{F^2_\phi}$. 
Note: if $1 < p < \infty$ and $A$ is compact on $F_\phi^p$ then:
Berezin transform and compactness

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$$\leq \limsup_{|w|\to\infty} \|A_kw\|_{F_p^\phi} \|k\|_{F_q^\phi} = 0.$$
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Note: if $1 < p < \infty$ and $A$ is compact on $\mathcal{F}^p_\phi$ then:

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Given a “nice” class \( X \) of functions on \( \mathbb{C}^n \), let \( T_p(X) \) be the \( F^p_\phi \) operator norm closure of the algebra generated by \( \{ T_f : f \in X \} \).
Is the converse true? No, not always.

Given a “nice” class $X$ of functions on $\mathbb{C}^n$, let $T_p(X)$ be the $F_p^\phi$ operator norm closure of the algebra generated by $\{T_f : f \in X\}$.

For the classical Fock space setting ($\phi(z) = |z|^2 / 8t$ for some $t > 0$) we have:
Main results for the classical Fock Space

Theorem 1 (Bauer/J.I. ’12, J.I., in preparation)

If $1 < p < \infty$ and $A$ is bounded on $F^p_t$ then $A$ is compact if and only if
$$\lim_{|w| \to \infty} (B(A))(w) = 0$$
and $A \in T_p(L^\infty(\mathbb{C}^n))$. 

Question: how much of Theorem 1 can be extended to arbitrary generalized Fock spaces?

Answer: remarkably, a lot of Theorem 1!
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Furthermore, any bounded $A$ is compact if and only if $A$ is in the $F^p_t$ operator norm closure of $\{T_f : f \in C^\infty_c(\mathbb{C}^n)\}$.
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- Answer: remarkably, a lot of Theorem 1!
Difficulties with Generalized Fock spaces

- Difficulties:
  - No explicit form for $K(z, w)$.
  - No explicit useful orthogonal basis (i.e. monomials!).
  - No invariance properties: $U_w f(z) := f(z - w)$ is NOT unitary.

In particular, we do not necessarily have a uniformly bounded family of operators $\{U_w\}_{w \in \mathbb{C}^n}$ on $F_p$ where $$(U_w^k_\eta(z)) = \Theta(\eta, w)^{k_\eta+w(z)} (1)$$ with $|\Theta(\cdot, \cdot)|$ bounded above and below on $\mathbb{C}^n \times \mathbb{C}^n$. 

Joshua Isralowitz Compactness of operators
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In particular, we do not necessarily have a uniformly bounded family of operators $\{U_w\}$ for $w \in \mathbb{C}^n$ on $F^n_{\varphi}$ where 

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- In particular, we do **not** necessarily have a uniformly bounded family of operators $\{U_w\}_{w \in \mathbb{C}^n}$ on $F^p_\phi$ where

$$
(U_w k_\eta)(z) = \Theta(\eta, w)k_{\eta+w}(z)
$$

with $|\Theta(\cdot, \cdot)|$ bounded above and below on $\mathbb{C}^n \times \mathbb{C}^n$. 
Let $\mathcal{SL}(\phi)$ be the class of operators $A$ such that $A$ is bounded on $F^q_\phi$ for some $2 \leq q < \infty$ and where

$$|\langle Ak_z, k_w \rangle^2_{F^2_\phi}| \leq \frac{C}{(1 + |z - w|)^{2n+\delta}}$$

for some $C > 0$ and $\delta > 0$ independent of $z, w \in \mathbb{C}^n$. 

Note: if $A \in \mathcal{SL}(\phi)$ then $A$ is bounded on $F^p_\phi$ for any $1 < p < \infty$ (J.I., in preparation).

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Note: \( T_f \in \mathcal{SL} (\phi) \) if \( f \in L^\infty (\mathbb{C}^n) \).
Main results for generalized Fock spaces

- Note: \( T_f \in \mathcal{SL}(\phi) \) if \( f \in L^\infty(\mathbb{C}^n) \). In particular there exists \( \epsilon > 0 \) independent of \( f \) and \( z, w \in \mathbb{C}^n \) where

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|\langle T_f k_z, k_w \rangle_{F_\phi^2}| \leq \|f\|_{L^\infty} e^{-\epsilon|z-w|}.
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- **Note:** $T_f \in \mathcal{SL}(\phi)$ if $f \in L^\infty(\mathbb{C}^n)$. In particular there exists $\epsilon > 0$ independent of $f$ and $z, w \in \mathbb{C}^n$ where

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**Theorem 2 (J.I., in preparation)**

Let $1 < p < \infty$ and let $A \in \mathcal{SL}(\phi)$. Then $A$ is compact on $F_\phi^p$ if and only if there exists $N = N(A)$ such that

$$\lim_{|w| \to \infty} \sup_{z \in B(w, N)} |\langle Ak_w, k_z \rangle_{F_\phi^2}| = 0.$$
Main results for generalized Fock spaces

Theorem 3 (J.I., in preparation)

Let $A$ be in the $F^2_\phi$ operator norm closure of $\mathcal{S}\mathcal{L}(\phi)$. Then $A$ is compact on $F^2_\phi$ if and only if there exists $N = N(A)$ such that

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Note: a slightly weaker version of Theorem 3 was (essentially) proven in Xia/Zheng '13 for the classical Fock space $F^2_t$. 

Question: what about the Berezin transform in $F^2_\phi$?!
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Lemma 4 (J.I., in preparation)

Let $A$ be any bounded operator on $F^2_\phi$. 
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Let A be any bounded operator on $F^2_\phi$. If there exists a uniformly bounded family of operators $\{U_w\}_{w \in \mathbb{C}^n}$ on $F^2_\phi$ as before, then the following are equivalent for any $N > 0$:

a) $\lim_{|w| \to \infty} |(B(A)(w))| = \lim_{|w| \to \infty} |\langle A_k w, k w \rangle|_{F^2_\phi} = 0$

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(b) $\lim_{|w|\to\infty} \sup_{z\in B(w,N)} |\langle Ak_w, k_z\rangle_{F^2_\phi}| = 0$. 

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Compactness of operators
Main results for generalized Fock spaces

- Necessary conditions for compactness:

Theorem 5 (J.I., in preparation)

Let $1 < p < \infty$. If $A$ is compact on $F_p^\infty$, then $A \in T_p(C_\infty c(C_n))$.

Furthermore, if $A$ is compact on $F_2^\infty$, then $A$ is in the $F_2^\infty$ operator norm closure of $\{T_f : f \in C_\infty c(C_n)\}$. 
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Furthermore, if $A$ is compact on $F^2_\phi$ then $A$ is in the $F^2_\phi$ operator norm closure of $\{T_f : f \in C^\infty_c(C^n)\}$. 
Open problems

Question 1:
Does Theorem 3 hold when $p \neq 2$?
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Can we get an estimate for the $F^2_\phi$ essential norm of operators in the $F^2_\phi$ operator norm closure of $SL(\phi)$?
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Question 1:
Does Theorem 3 hold when $p \neq 2$?

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Do Theorems 2 and 3 hold when the Berezin transform vanishes at infinity?

Question 3:
Can we get an estimate for the $F^2_\phi$ essential norm of operators in the $F^2_\phi$ operator norm closure of $SL(\phi)$? What about just for operators in $SL(\phi)$, or even just if $A$ is a Toeplitz operator?
Note: given the existence of a family of operators \( \{U_w\} \) with \( w \in \mathbb{C}^n \) on \( F_2^\phi \) as above, we have
\[
\|A\|_{e} \approx \limsup_{|w| \to \infty} \|A_k w\|_{F_2^\phi}
\]
for all \( A \) in the \( F_2^\phi \) operator norm closure of \( SL(\phi) \) (J.I., in preparation, see also Mitkovski/Wick, preprint).

Question 4: Is every compact operator on \( F_p^\phi \) necessarily in the \( F_p^\phi \) operator norm closure of \( \{T_f : f \in C^\infty c(\mathbb{C}^n)\} \)?
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Note: given the existence of a family of operators \( \{U_w\}_{w \in \mathbb{C}^n} \) on \( F_\phi^2 \) as above, we have

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Question 4:
Is every compact operator on \( F_\phi^p \) necessarily in the \( F_\phi^p \) operator norm closure of \( \{T_f : f \in C_\infty^\infty(\mathbb{C}^n)\} \)?
Question 5:

Does the $F^p_\phi$ operator norm closure of $SL(\phi)$ coincide with $T_p(L^\infty(\mathbb{C}^n))$?
Question 5:

Does the $F_p^\phi$ operator norm closure of $\mathcal{SL}(\phi)$ coincide with $T_p(L^\infty(\mathbb{C}^n))$? What about even on the classical Fock space?
Question 5:

Does the $F^p_\phi$ operator norm closure of $\mathcal{SL}(\phi)$ coincide with $T_p(L^\infty(\mathbb{C}^n))$? What about even on the classical Fock space?

- Note: a useful tool developed by D. Suarez for studying related questions on the Bergman space of the disk and ball is the \textbf{k-Berezin transform}. 
Question 6:
Can one construct a k-Berezin transform on the classical Fock space?
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Can one construct a k-Berezin transform on the classical Fock space?

Thank You!