Self-commutators of Toeplitz operators and isoperimetric inequalities

Steve Bell\textsuperscript{1}, Tim Ferguson\textsuperscript{2}, Erik Lundberg\textsuperscript{1}

\textsuperscript{1}Department of Mathematics  
Purdue University

\textsuperscript{2}Department of Mathematics  
Vanderbilt University

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Putnam’s Inequality

Let $T$ be a hyponormal operator, i.e. $[T^*, T] = T^* T - TT^* \geq 0$. In 1970, Putnam proved

$$\|[T^*, T]\| \leq \frac{\text{Area}(\text{sp}(T))}{\pi}$$

where sp denotes the spectrum.

In particular, this inequality applies to Toeplitz operators with analytic symbol $\phi$.

$T_\phi(f) = \phi f$ and $T^{*\phi}(f) = P(\overline{\phi} f)$. 
Khavinson’s Theorem

In 1984, Khavinson proved the following. Let $E_2(G)$ denote the Smirnov space associated to a domain $G$ bounded by a finite number of smooth Jordan arcs. Let $\phi$ be analytic in a neighborhood of $\overline{G}$. Then

$$
\|[T_\phi^*, T_\phi]\| \geq \frac{4 \text{Area}^2(\text{sp}(T_\phi))}{\|\phi\|^2_{E_2(G)} \text{perim}(G)}
$$

Here $T_\phi(f) = \phi f$ and $T_\phi^*(f) = P(\overline{\phi} f)$. 

Khavinson noted the following. If one takes $\phi(z) = z$ then
\[ \|\phi'\|^2_{E_2(G)} = \text{perim}(G) \] and $\text{sp}(T_z) = \overline{G}$. Thus combining his result with Putnam’s yields
\[ \frac{4 \text{Area}^2(G)}{\|\phi'\|^2_{E_2(G)} \text{perim}(G)} \leq \|[T^*_\phi, T_\phi]\| \leq \frac{\text{Area}(G)}{\pi}. \]

Upon rearrangement, this becomes the classical isoperimetric inequality
\[ \text{perim}(G)^2 \geq 4\pi \text{Area}(G). \]
We now consider a similar question in the Bergman space \( A^2(G) \). Suppose that \( G \) is a bounded \( C^1 \) domain in the plane, and \( \phi \) is analytic and univalent in \( G \), as well as sufficiently regular on \( \partial G \). Then we have the following bound

\[
\|[T^*_\phi, T_\phi]\| \geq \frac{16\pi}{\lambda^2 \text{Area}(\phi(G))}
\]

where \( \lambda \) is the first eigenvalue of the Laplacian on \( \phi(G) \).
An extension

In fact, we can generalize this to the case where $\phi$ is only locally univalent.

**Theorem 1 (Bell, F., Lundberg)**

Let $G_\phi$ be the Riemann surface formed by sheets over $\phi(G)$, so that $\phi$ is one-to-one from $G$ to $\phi(G)$. Then

$$\|[T_\phi^*, T_\phi]\| \geq \frac{16\pi}{\lambda^2 \text{Area}(G_\phi)}$$

where $\lambda$ is the first eigenvalue of the Laplacian on $G_\phi$. 
Torsional Rigidity

Recall that the first eigenfunction $\psi$ of the Laplacian for a given region maximizes the quotient $\|\psi\|_2/\|\nabla \psi\|_2$ among all functions zero on the boundary of the given region.

If we instead choose $\psi$ so that it maximizes $\|\psi\|_1/\|\nabla \psi\|_2$, we are led to the concept of torsional rigidity.

Torsional rigidity is the maximum of

$$\left( \frac{2\|\psi\|_1}{\|\nabla \psi\|_2} \right)^2$$

Of all domains with a given area, a circle has the largest torsional rigidity, and the smallest principal frequency.
Theorem about torsional rigidity

**Theorem 2 (Bell, F., Lundberg)**

Let $\phi$ be locally univalent as before. Then

$$\| [T^*_\phi, T_\phi] \| \geq \frac{\rho}{\text{Area}(G_\phi)}$$

Here $\rho$ is the torsional rigidity of $G_\phi$.

Our methods more naturally lead to this theorem than the theorem involving the first eigenvalue of the Laplacian.

There is reason to suspect this bound is sharp. We know it is sharp when $G = \mathbb{D}$ and $\phi = z$. 
Bounds for $\lambda$ and $\rho$

Take $\phi = z$. Combine Putnam’s inequality and our results:

$$\lambda \geq \frac{4\pi}{\text{Area}(G)}$$

$$\rho \leq \frac{\text{Area}(G)^2}{\pi}$$

where $G$ is any domain. The sharp inequalities are:

$$\lambda \geq \frac{j^2\pi}{\text{Area}(G)}$$

$$\rho \leq \frac{\text{Area}(G)^2}{2\pi}$$

(the Fraber-Krahn Theorem, and the Saint-Venant theorem).

Here $j \approx 2.4$ is the first positive zero of the Bessel function $J_0$. 
We expect that our bound for the norm of the commutator in terms of $\rho$ is sharp. However, the bound for $\rho$ we obtain from this and Putnam’s theorem is off by a factor of two.

Thus, it is tempting to conjecture that

$$\|[[T^*_\phi, T_\phi]] \leq \frac{\text{Area}(G_\phi)}{2\pi}$$

for Toeplitz operators with analytic symbols acting on the Bergman space of $G$. This would be an improvement of the original Putnam inequality by a factor of 2 for this case.

Putnam’s inequality is sharp in general, but perhaps not for this special case.
Let $\phi$ be univalent. Note that $\rho_W$ increases with the domain $W$. If we use this fact for $W$ being the largest disc contained in $G$, and use Theorem 2 we obtain

$$\|[[T^*_{\phi}, T_{\phi}]\| \geq \frac{\pi R^4_l}{2 \text{Area}(\phi(G))}$$

where $R_l$ is the radius of the largest disc contained in $\phi(G)$.

We have used the fact that, for a disc of radius $r$, $\rho = \frac{\pi r^4}{2}$.
Some ideas from the proof

The ideas on this slide follow Khavinson’s original proof.

- Since $[T^*_\phi, T_\phi]$ is normal, $\|[T^*_\phi, T_\phi]\| = \sup_{h \in A^2, \|h\|=1} \langle [T^*_\phi, T_\phi] h, h \rangle$

- Use the fact that $T^*_\phi(f) = P(\overline{\phi f})$ to conclude that

$$\|[T^*_\phi, T_\phi]\| = \sup_{\|h\|=1} \text{dist}(\overline{\phi h}, A^2(G))^2 = \sup_{\|h\|=1} \{ \inf_{f \in A^2} \|\overline{\phi h} - f\|_2 \}^2$$

- Use Hahn-Banach duality to conclude that

$$\|[T^*_\phi, T_\phi]\| = \sup_{\|h\|=1} \inf_{\|f\| \in A^2} \sup_{\|g\|=1} \left| \int_G (\overline{\phi h - f}) \overline{g} \, dA \right| .$$

- We want to choose $g$ so that the integral does not depend on $f$, and make an appropriate choice of $h$. 
More ideas from the proof

- Take $h = \phi' / \|\phi'\|_2$.
- We want to choose $g = \partial_z \psi / \|\partial_z \psi\|_2$, where $\psi$ is sufficiently regular and vanishes on $\partial G$.
- By Cauchy-Green (or Havin’s lemma), we see that this makes the integral in the last slide independent of $f$.
- We choose $\psi(z) = \psi_W(\phi(z))$, where $\psi_W$ minimizes either the Raleigh quotient for $W$, or the torsional rigidity quotient for $W$.
- This choice leads to our stated inequalities.
Illustration in the case $\phi = z$.

- We choose $h = 1 / \text{Area}(G)$.
- Let $\psi$ be the function maximizing torsional rigidity, (or minimizing the least eigenvalue of the Laplacian on $G$).
- Choose $g = \partial_z \psi / \| \partial_z \psi \|_2$.

\[
\| [T^*_\phi, T_\phi] \| \geq \frac{1}{\| \partial_z \psi \|_2 \text{Area}(G)} \inf_{\| f \| \in A^2} \left| \int_G (\bar{z} - f) \overline{\partial_z \psi} \, dA \right|
\]

\[
= \frac{1}{\| \partial_z \psi \|_2 \text{Area}(G)} \left| \int_G \bar{z} \overline{\partial_z \psi} \, dA \right|
\]

- Apply Cauchy-Green and the fact that $\psi$ is nonnegative:

\[
\frac{1}{\| \partial_z \psi \|_2 \text{Area}(G)} \left| \int_G \bar{\psi} \, dA \right| = \frac{1}{\| \partial_z \psi \|_2 \text{Area}(G)} \int_G |\psi| \, dA = \frac{\rho_G}{\text{Area}(G)}
\]
Double Quadrature Domains

A bounded domain $G \subset \mathbb{C}$ is called an *area quadrature domain* if it admits a formula expressing the area integral of any function $f$ analytic and integrable in $G$ as a finite sum of weighted point evaluations of the function and its derivatives. i.e.

$$\int_{\Omega} f \, dA = \sum_{m=1}^{N} \sum_{k=0}^{n_m} a_{m,k} f^{(k)}(z_m),$$

where $z_m$ are distinct points in $\Omega$ and $a_{m,k}$ are constants (possibly complex) independent of $f$.

In order to define *arclength quadrature domains*, such a formula is prescribed to hold for integration over the boundary with respect to arclength instead of area measure.

A double quadrature domain is both an area and an arclength quadrature domain.
Figure: The domain $G$ when $\varepsilon = 0.5$. 

Picture of a Double Quadrature Domain
Bounds for $[T_z^*, T_z]$ on the Domain

Figure: The norm of $[T_z^*, T_z]$ (solid), and the lower (dashed) and upper (dotted) bounds plotted against $\varepsilon$. 