Carleson and Vanishing Carleson Measures on Radial Trees

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Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $m$ the normalized Lebesgue measure on $\partial \mathbb{D}$. For $p > 0$, let $L^p(m)$ be the space of functions $f$ on $\partial \mathbb{D}$ such that

$$\|f\|_{L^p(m)} = \left( \int_{\partial \mathbb{D}} |f(\zeta)|^p \, dm(\zeta) \right)^{1/p} < \infty.$$

For $f \in L^p(m)$, the Poisson integral of $f$ defined as

$$Pf(z) = \int_{\partial \mathbb{D}} P_\zeta(z)f(\zeta) \, dm(\zeta) \quad (z \in \mathbb{D})$$

is harmonic on $\mathbb{D}$, where $P_\zeta(z) = \frac{1 - |z|^2}{|\zeta - z|^2}$ is the Poisson kernel at $\zeta$.

Given $\theta_0 \in \mathbb{R}$ and $h \in (0, 1)$, let

$$S_{\theta_0, h} := \{re^{i\theta} : 1 - h \leq r < 1, |\theta - \theta_0| \leq h/2\}.$$
Let $\sigma$ be a positive measure on $\mathbb{D}$. TFAE:

(a) $\exists C > 0$ such that $\forall \theta_0 \in \mathbb{R}, \forall h \in (0, 1), \sigma(S_{\theta_0}h) \leq C h$.
(b) $\forall p > 1$, the Poisson operator $P : L^p(m) \to L^p(\sigma)$ is bounded.
(c) $\exists C > 0$ such that $\forall h$ harmonic, $\forall \lambda > 0$,

$$\sigma(\{ v \in T : |h(v)| > \lambda \}) \leq C m(\{ \zeta \in \partial \mathbb{D} : h^*(\zeta) > \lambda \}),$$

where $h^*(\zeta) = \sup_{0 < r < 1} |h(r\zeta)|$.

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Let $\sigma$ be a positive measure on $\mathbb{D}$. TFAE:

(a) $\exists C > 0$ such that $\forall \theta_0 \in \mathbb{R}, \forall h \in (0, 1), \sigma(S_{\theta_0}h) \leq C h$.
(b) $\forall p > 0$, the identity operator from $H^p$ to $L^p(\sigma)$ is bounded, where

$$\|f\|_{H^p}^p := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$
Given $s \geq 1$, a positive measure $\sigma$ on $\mathbb{D}$ is an $s$-Carleson measure if $\exists C > 0$ such that $\forall \theta_0 \in \mathbb{R}$, $\forall h \in (0, 1)$, $\sigma(S_{\theta_0,h}) \leq C h^s$.


Let $s \geq 1$ and $\sigma$ a positive measure on $\mathbb{D}$. TFAE:

(a) $\sigma$ is an $s$-Carleson measure.
(b) $\forall p > 0$, the identity operator from $H^p$ to $L^{sp}(\sigma)$ is bounded.

$\sigma$ is an vanishing $s$-Carleson measure iff

$$\limsup_{h \to 0} \sup_{\theta \in \mathbb{R}} \frac{\sigma(S_{\theta,h})}{h^s} = 0.$$ 


$\sigma$ is a vanishing 1-Carleson measure iff $\forall p > 1$, the Poisson operator $P : L^p(m) \to L^p(\sigma)$ is compact.
For $z \in \mathbb{D}$ and $w \in \overline{\mathbb{D}}$, the extended Poisson kernel at $(z, w)$ is defined by

$$P(z, w) = \frac{1 - |z|^2}{|1 - \overline{z}w|^2}.$$ 

$P(z, w) = P_w(z)$ when $|w| = 1$.

**Other characterizations**

Given a finite measure $\sigma$ on $\mathbb{D}$.

1. $\sigma$ is a 1-Carleson measure iff

$$\sup_{|z| \in \mathbb{D}} \int_{\mathbb{D}} P(z, w) \, d\sigma(w) < \infty.$$ 

2. $\sigma$ is vanishing 1-Carleson iff

$$\lim_{|z| \to 1} \int_{\mathbb{D}} P(z, w) \, d\sigma(w) = 0.$$
Objectives

- Define Carleson squares on a radial tree $T$.
- Define the notion of extended Poisson kernel on trees endowed with a Green function.
- Develop the notions of $s$-Carleson and vanishing $s$-Carleson measure on $T$.
- Define the harmonic Hardy spaces $\mathcal{H}^p$ on $T$.
- Obtain a version of Duren’s and Power’s theorems for $\mathcal{H}^p$ as well as weak-type results for $p = 1$. 
- A tree is a locally finite, connected and simply-connected graph which we identify with the set of its vertices.

- Two vertices $v$ and $w$ are called neighbors (and we write $v \sim w$) if there exists an edge connecting them.

- A path is a sequence of vertices $[v_0, v_1, \ldots]$ such that $\forall k$, $v_k \sim v_{k+1}$ and $v_k \neq v_{k+1}$.

- Denote by $[v, w]$ the unique path connecting $v$ to $w$.

- Fixing a root $e$ and a vertex $w$, $v$ is called a descendant of $w$ if $w \in [e, v]$. The parent of $v \neq e$ is the neighbor $v^-$ of $v$ closest to $e$.

- The length of a path $[u, v]$ is the number $d(u, v)$ of its edges. The length of a vertex $v$ is $|v| = d(e, v)$. 
Carleson squares on a tree $T$

Given $v \in T$, the sector relative to $v$ is

$$S_v = \{v\} \cup \{w \in T : w \text{ is a descendant of } v\}.$$ 

$\partial T$ is the set of infinite paths $\omega = [e = \omega_0, \omega_1, \omega_2, \ldots)$. Under the topology generated by the sets

$$I_v = \{\omega \in \partial T : v \in [e, \omega)\},$$

$\partial T$ is compact and $T \cup \partial T$ is a compactification of $T$.

Properties

- $\partial T = I_e$.
- If $v, w \in T$, either $S_v \subset S_w$, or $S_w \subset S_v$, or $S_v \cap S_w = \emptyset$.
- $\forall v \in T$, $\partial S_v = I_v$.
- $\forall n \in \mathbb{N}$, $\partial T = \bigcup_{|v|=n} I_v$.  

Given \( q \in \mathbb{N} \), a tree is homogeneous of degree \( q + 1 \) if all its vertices have \( q + 1 \) neighbors.

The number of vertices of length \( n \) is \( c_n = \begin{cases} 1 & \text{if } n = 0, \\ (q + 1)q^{n-1} & \text{if } n \geq 1. \end{cases} \)

A nearest-neighbor transition probability is a function \( p : T \times T \to [0, 1] \) such that \( \forall v \in T, \sum_{w \sim v} p(v, w) = 1 \) and \( p(v, u) > 0 \) iff \( v \sim u \).

Laplace operator

Given \( f : T \to \mathbb{R} \), define

\[
\Delta f(v) = \sum_{w \sim v} p(v, w)f(w) - f(v), \quad v \in T.
\]

\( f \) is called harmonic (resp. superharmonic, subharmonic) if \( \Delta f = 0 \) (resp. \( \Delta f \leq 0, \Delta f \geq 0 \)).
For $u, v \in T$, let $F(u, v)$ be the probability that the associated random walk starting at $u$ hits $v$ in positive time.

Green’s function

For $u, v \in T$, let $G(u, v)$ the expected number of visits to vertex $v$ by the random walk starting at $u$. Then

$$G(v, w) = \begin{cases} (1 - F(v, v))^{-1} & \text{if } v = w, \\ F(v, w)G(w, w) & \text{if } v \neq w. \end{cases}$$

If $[v_0, v_1, \ldots, v_n]$ is the path from $v_0$ to $v_n$, then

$$F(v_0, v_n) = \prod_{k=0}^{n-1} F(v_k, v_{k+1}).$$

$T$ is called transient if $G(u, v) < \infty$ for some (hence all $u, v \in T$). A non-transient tree is called recurrent.
TFAE:

1. $T$ is transient.
2. $\exists v \in T$ such that $F(v, v) < 1$.
3. $\forall v \in T$, $F(v, v) < 1$.
4. $\exists f$ positive superharmonic, nonharmonic on $T$.

From now on we assume $T$ transient.

**Poisson kernel**

For $v \in T$, $\omega \in \partial T$, let $P_\omega(v)$ denote the value of the Poisson kernel at $(\omega, v)$. Every positive harmonic function on $T$ can be written as $P_\mu(\cdot) := \int_{\partial T} P_\omega(\cdot) \, d\mu(\omega)$ for a unique Borel measure $\mu$ on $\partial T$. 
Relation between the Poisson kernel, $G$, and $F$

\[ P_\omega(v) = \frac{G(v, v \wedge \omega)}{G(e, v \wedge \omega)} = \begin{cases} \frac{F(v, v \wedge \omega)}{F(e, v \wedge \omega)} & \text{if } v \wedge \omega \neq v, \\ 1 & \text{if } v \wedge \omega = v \neq e, \\ \frac{1}{F(e, v)} & \text{if } v = e, \end{cases} \]

where $v \wedge \omega$ is the closest vertex $\omega_k$ to $v$.

Homogeneous isotropic case: $p(v, w) = \frac{1}{q+1}$ for $v \sim w$

Given $v \in T$, $\omega \in \partial T$, the Poisson kernel is given by

\[ P_\omega(v) = q^{2|v \wedge \omega| - |v|}. \]
A tree is radial if for all \( v \sim w \), \( p(v, w) \) depends only on \( |v| \) and \( |w| \). If so, the degree of a vertex \( v \) depends only on \( |v| \).

For all \( k \geq 0 \), let \( q_k \) be the number of forward neighbors of any vertex of length \( k \). So for \( n \in \mathbb{N} \), \( c_n = \prod_{k=0}^{n-1} q_k \).

**Lebesgue measure on \( \partial T \)**

Let \( m \) be the probability measure on \( \partial T \) such that

\[
m(l_v) = \frac{1}{c_n}, \text{ for } |v| = n \geq 1.
\]

If \( \mu \) is absolutely continuous with respect to \( m \) with density \( f \) (i.e. \( d\mu(\omega) = f(\omega)dm(\omega) \)), we write \( Pf \) instead of \( P\mu \).
In addition to assuming $T$ radial, we assume $\exists q, C_1, C_2 > 0$ and $\delta_1, r \in (0, 1)$ such that:

$A_1$: $2 \leq q_j \leq q$ for each $j$;

$A_2$: $P_\omega(v) \leq C_1 m(l_v \wedge \omega)^{-1} |v| - |v \wedge \omega|$, for all $v \in T, \omega \in \partial T$;

$A_3$: $P_\omega(v) \geq C_2 m(l_v)^{-1}$, for all $v \in T, \omega \in l_v$;

$A_4$: $F(v, v^-) \leq 1 - \delta_1$, for all $v \neq e$. 
Extended Poisson kernel

\[ \mathcal{P}(v, w) = \begin{cases} 
1 & \text{if } v, w \in T, v = e \\
F(e, v \wedge w)^{-1} & \text{if } v, w \in T, v \wedge w = v \neq e, \\
\frac{F(v, v \wedge w)}{F(e, v \wedge w)} & \text{if } v, w \in T, v \wedge w \neq v, \\
P_\omega(v) & \text{if } v \in T, w = \omega \in \partial T.
\end{cases} \]

Relation with the Poisson kernel

Let \( v, w \in T \), and \( \omega \in I_w \). Then

\[ \mathcal{P}(v, w) = \begin{cases} 
P_\omega(v) & \text{if } v \wedge w = v \wedge \omega, \\
P_\omega(v)F(w, v \wedge \omega)F(v \wedge \omega, w) & \text{if } v \wedge w \neq v \wedge \omega.
\end{cases} \]

In particular, \( \mathcal{P}(v, w) \leq P_\omega(v) \). If \( v \in \{w\} \cup (T \setminus S_w) \), then

\[ C_2 m(I_v)^{-1} \leq \mathcal{P}(v, w) \leq C_1 m(I_{v \wedge w})^{-1} r^{|v| - |v \wedge w|} \]
Harmonic Hardy Spaces

For $1 \leq p < \infty$, let $L^p(m)$ be the set of functions $f$ on $\partial T$ such that

$$
\|f\|_{L^p(m)}^p = \int_{\partial T} |f(\omega)|^p \, dm(\omega) < \infty.
$$

Taibleson (1987) defined the harmonic Hardy spaces $H^p$ on $T$ in terms of certain functions in $L^p(m)$.

Given $n \in \mathbb{N}$, $1 \leq p < \infty$ and a function $h$ on $T$, let

$$
M_p(h, n) = \sum_{|v|=n} |h(v)|^p c_{|v|} = m(l_v) \sum_{|v|=n} |h(v)|^p.
$$

Alternative definition of $H^p$

Let $h$ be harmonic on $T$. Then $h \in H^p$ iff

$$
\|h\|_{H^p}^p := \sup_n M_p(h, n) < \infty.
$$
Radial maximal function

For \( h \) harmonic on \( T \), let

\[
h^*(\omega) = \sup_{n \in \mathbb{N}} |h(\omega_n)|, \quad \text{for } \omega \in \partial T.
\]

Characterization of functions in \( \mathcal{H}^p \)

Given \( h \) harmonic and \( 1 < p < \infty \), TFAE:

(a) \( h \in \mathcal{H}^p \).
(b) \( \exists f \in L^p(m) \) such that \( h = Pf \).
(c) \( h^* \in L^p(m) \).
(d) \( |h|^p \) has a harmonic minorant.

Theorem

\( \exists C > 0 \) such that \( \forall f \in L^p(m), \)

\[
C\|f\|_{L^p(m)} \leq \|Pf\|_{\mathcal{H}^p} \leq \|f\|_{L^p(m)}.
\]
Hardy-Littlewood maximal function

\[
Mf(\omega) = \sup_{\{v \in T : \omega \in I_v\}} \frac{1}{m(I_v)} \int_{I_v} |f(\tau)| \, dm(\tau), \quad f \in L^1(m).
\]

For \(1 \leq p < \infty\), and \(f \in L^p(m)\), \(Mf < \infty\). Moreover,

1. \(\forall f \in L^1(m), \forall \lambda > 0, (Pf)^\ast \leq \frac{C_1}{1-s} Mf\), and

\[
m(\{\omega : Mf(\omega) > \lambda\}) \leq \frac{1}{\lambda} \|f\|_{L^1(m)}.
\]

2. If \(p > 1\), \(\exists C > 0\) such that \(\forall f \in L^p(m), \|Mf\|_{L^p(m)} \leq C \|f\|_{L^p(m)}\).

Let \(s \geq 1\) and \(\sigma\) a nonnegative measure on \(T\).
\(\sigma\) is an \(s\)-Carleson measure iff

\[
\sigma(S_v) = O(m(I_v)^s) \quad \text{as} \quad |v| \to \infty.
\]

\(\sigma\) is a vanishing \(s\)-Carleson measure iff

\[
\sigma(S_v) = o(m(I_v)^s) \quad \text{as} \quad |v| \to \infty.
\]
Let $1 \leq s < \infty$, and $\sigma$ a finite measure on $T$. TFAE:

1. $\sigma$ is an $s$-Carleson measure.
2. $\sup_{v \in T} \sum_{w \in T} \mathcal{P}(v, w)^s \sigma(\{w\}) < \infty$.
3. $\forall p > 1, \forall f \in L^p(m), Pf \in L^{sp}(\sigma)$.
4. $\forall p > 1, P : L^p(m) \to L^{sp}(\sigma)$ is bounded.
5. $\exists C > 0$ such that $\forall h$ harmonic, $\forall \lambda > 0$,
   $$\sigma(\{v \in T : |h(v)| > \lambda\}) \leq C (m\{\omega : h^*(\omega) > \lambda\})^s.$$
6. $\exists C > 0$ such that $\forall f \in L^1(m), \forall \lambda > 0$,
   $$\sigma(\{v \in T : |Pf(v)| > \lambda\}) \leq \frac{C}{\lambda^s} \|f\|^s_{L^1(m)}.$$
Proof. (1) $\iff$ (2) is based on the extended Poisson kernel estimates. 

(1) $\Rightarrow$ (5): Let $h$ be harmonic. For $\lambda > 0$, define $A = \{ v : |h(v)| > \lambda \}$. Then $\exists \hat{A} \subseteq A$ such that $\bigcup_{v \in \hat{A}} S_v = \bigcup_{v \in A} S_v$, $\bigcup_{v \in \hat{A}} l_v = \bigcup_{v \in A} l_v$, and for $v, w \in \hat{A}$ with $v \neq w$, $S_v \cap S_w = \emptyset$ and $l_v \cap l_w = \emptyset$. Since for $\omega \in l_v$ with $v \in A$, $h^*(\omega) \geq |h(v)| > \lambda$,

$$
\sigma(\{ v : |h(v)| > \lambda \}) \leq \sigma\left( \bigcup_{v \in \hat{A}} S_v \right) = \sum_{v \in \hat{A}} \sigma(S_v) \leq C \sum_{v \in \hat{A}} m(l_v)^s
$$

$$
\leq C \left( \sum_{v \in \hat{A}} m(l_v) \right)^s = C \left( m\left( \bigcup_{v \in \hat{A}} l_v \right) \right)^s
$$

$$
= C \left( m\left( \bigcup_{v \in A} l_v \right) \right)^s \leq C \left( m\{ \omega : h^*(\omega) > \lambda \} \right)^s.
$$
(5) ⇒ (4): Let \( f \in L^p(m) \) and \( h = Pf \). Then

\[
\|h\|_{L^p(\sigma)}^{sp} = \int_0^\infty p\lambda^{p-1}\sigma\{v : |h(v)|^s > \lambda\} \, d\lambda
\]

\[
= \int_0^\infty p\lambda^{p-1}\sigma\{v : |h(v)| > \lambda^{1/s}\} \, d\lambda
\]

\[
\leq C \int_0^\infty p\lambda^{p-1}(m\{\omega : h^*(\omega) > \lambda^{1/s}\})^s \, d\lambda
\]

\[
= C \int_0^\infty sp\lambda^{sp-1}(m\{\omega : h^*(\omega) > \lambda\})^s \, d\lambda
\]

\[
\leq C \int_0^\infty sp\lambda^{sp-1}(m\{\omega : Mf(\omega) > \lambda\})^s \, d\lambda
\]

\[
\leq C\|Mf\|_{L^p(m)}^{sp}
\]

\[
\leq C\|f\|_{L^p(m)}^{sp}.
\]
Let $\sigma$ be a finite measure on $T$ and $s \geq 1$. TFAE:

1. $\sigma$ is a vanishing $s$-Carleson measure.

2. For $1 < p < \infty$, $P : L^p(m) \to L^{sp}(\sigma)$ is compact.

3. \[
\lim_{|v| \to \infty} \sum_{w \in T} P(v, w)^s \sigma(\{w\}) = 0.
\]

4. $\forall \{f_n\}$ in $L^1(m)$ converging to 0 weakly and $\forall \lambda > 0$,
   \[
   \lim_{n \to \infty} \frac{\sigma(\{v \in T : |Pf_n(v)| > \lambda\})}{\|f_n\|_{L^1(m)}^s} = 0.
   \]

5. $\forall \{h_n\}$, $h_n$ harmonic and converging to 0 pointwise, and $\forall \lambda > 0$,
   \[
   \lim_{n \to \infty} \frac{\sigma(\{v \in T : |h_n(v)| > \lambda\})}{m(\{\omega : h_n^*(\omega) > \lambda\})^s} = 0.
   \]

6. $\forall \{f_n\}$ in $L^1(m)$ converging to 0 weakly and $\forall \lambda > 0$,
   \[
   \lim_{n \to \infty} \frac{\sigma(\{v \in T : |Pf_n(v)| > \lambda\})}{m(\{\omega : (Pf_n)^*(\omega) > \lambda\})^s} = 0.
   \]
Let $\sigma$ be a finite positive measure on $\mathbb{D}$, $p \geq 1$ and $s \geq 1$. TFAE:

1. $\sigma$ is a $2s$-Carleson measure.

2. $\exists C > 0$ such that $\forall f$ positive subharmonic on $\mathbb{D}$,

$$
\|f\|_{L^{sp}(\sigma)} \leq C\|f\|_{L^p(dA)}.
$$

In a homogeneous isotropic tree case, we can prove Hastings’ Theorem for $s = 1$ where we take $dA(\{v\}) = q^{-2|v|}$. This prompted us to look at the following questions:

1. Does Hastings’ Theorem hold if $dA$ is replaced by a more general radial measure?

2. If so, what conditions characterize such measures?

3. Can the class of subharmonic functions be replaced by the class of positive harmonic functions?