Wolff’s Ideal Problem in the Multiplier Algebra on Dirichlet Space

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1. Notations
   - Carleson’s Corona Theorem
   - Wolff’s Ideal Problem in $H_\infty(\mathbb{D})$

2. Wolff’s Ideal Problem in $\mathcal{M}(\mathcal{D})$
   - Outline of the Proof

3. Further Discussion
Introduction

- $H^\infty(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{C} \text{ such that } f \text{ is analytic on } \mathbb{D} \text{ with } \|f\|_{H^\infty(\mathbb{D})} = \sup_{z \in \mathbb{D}} |f(z)| < \infty \}.$

Hardy Space:
$H^2(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{C} \text{ such that } f \text{ is analytic on } \mathbb{D},$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \|f\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$ 

Dirichlet Space:
$D = \{ f : \mathbb{D} \to \mathbb{C} \text{ such that } f \text{ is analytic on } \mathbb{D},$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \|f\|^2_{B(D)} = \sum_{n=0}^{\infty} (n+1) |a_n|^2 < \infty.$$
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- Dirichlet Space:
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**Multiplier Algebra of Dirichlet space:**

\[ \mathcal{M}(D) = \{ \phi \in D : \phi f \in D \text{ for all } f \in D \} . \]

- Reproducing Kernel for the Dirichlet space is
  \[ k_w(z) = \frac{1}{zw} \log \left( \frac{1}{1 - zw} \right) . \]
  We can see that
  \[ \frac{1}{k_w(z)} = 1 - \sum_{n=1}^{\infty} c_n (zw)^n, \quad c_n > 0 \text{ for all } n. \]
  Hence, the Dirichlet space is the Reproducing Kernel Hilbert space with “one - positive square”.

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  \[ M_\phi(f) = \phi f \text{ for all } f \in D. \]
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\[ M_\phi(f) = \phi f \text{ for all } f \in \mathcal{D}. \]
If $\mathcal{H}(\Omega)$ has 1-positive square. Then the Commutant Lifting Theorem holds for $\mathcal{M}(\mathcal{H}(\Omega))$. (Sarason, Sz. Nazy-Foias, Rosenblum-Rovnyak,...Ball-Trent-Vinnikov etc.)

This gives a “Douglas” type lemma. (Leech, Schubert, Sz. Nazy-Foias, Arveson, Rosenblum, Agler-McCarthy,....)

Let $\mathcal{H}(\Omega)$ has a 1- positive square.
Then,
$$A, B \in \mathcal{M}(\mathcal{H}(\Omega) \otimes l^2) \text{ with } AA^* \geq BB^*$$
$$\implies \exists C \in \mathcal{M}(\mathcal{H}(\Omega) \otimes l^2) \text{ with } AC = B \text{ and } \|C\| \leq 1.$$
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It’s well known that $\mathcal{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D})$ but $\mathcal{M}(\mathcal{D}) \subsetneq H^\infty(\mathbb{D})$ (e.g., $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ is in $H^\infty(\mathbb{D})$ but is not in $\mathcal{D}$ and so neither in $\mathcal{M}(\mathcal{D})$). Thus, $\mathcal{M}(\mathcal{D})) \subsetneq H^\infty(\mathbb{D}) \cap \mathcal{D}$.

We will use the other equivalent norms for smooth functions in $\mathcal{D}$ as follows,

$$\|f\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{\mathbb{D}} |f'(z)|^2 dA(z)$$

and

$$\|f\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} d\sigma d\sigma$$
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Notations

- We use \( \mathcal{H}D \) to denote the harmonic Dirichlet space (restricted to the boundary of \( D \)). The functions in \( D \) have only positive coefficients where as the functions in \( \mathcal{H}D \) may have negative fourier coefficients too. Again, if \( f \) is smooth on \( \partial D \), the boundary of the unit disk, then

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- If \( h \in \mathcal{M}(D) \), then \( h \in \mathcal{M}(\mathcal{H}D) \) and

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\|h\|_{B(\mathcal{H}D)} \leq \sqrt{20} \|h\|_{B(D)}.
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3. Further Discussion
Carleson’s Corona Theorem

In 1962, Carleson proved his famous “Corona Theorem" characterizing when a finitely generated ideal in $H^\infty(\mathbb{D})$ is actually all of $H^\infty(\mathbb{D})$.

**Theorem**

If $f_1, f_2, ..., f_n$ are bounded analytic functions in the open unit disk $\mathbb{D}$ such that

$$|f_1(z)| + ... + |f_n(z)| \geq \delta > 0, \quad (0) \text{ for all } z \in \mathbb{D},$$

Then there exist bounded analytic functions $g_1, g_2, ..., g_n$ such that

$$f_1(z)g_1(z) + f_2(z)g_2(z) + ... + f_n(z)g_n(z) = 1 \text{ for all } z \in \mathbb{D}.$$
In other words, (0) implies that $1 \in \mathcal{I} \left( \{f_j\}_{j=1}^n \right)$, ideal generated by $\{f_j\}_{j=1}^n$

Now, a natural question arises here: what happens to the result if we replace the uniform lower bound by any bounded analytic function in (0)?

That means: Let $f_1, f_2, \ldots, f_n$ be $H^\infty(D)$ functions, and suppose $h \in H^\infty(D)$ satisfies

$$
(1) \quad |h(z)| \leq |f_1(z)| + \ldots + |f_n(z)| \text{ for all } z \in \mathbb{D}.
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Then the question is, does (1) imply $h \in \mathcal{I} \left( \{f_j\}_{j=1}^n \right)$?

The answer is No! [Garnett (Bounded Analytic Functions)]
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Wolff’s Ideal Problem in $H^\infty(D)$

However, T. Wolff [1980] has proved that $h^3 \in \mathcal{I} \left( \{f_j\}_{j=1}^n \right)$

**Theorem**

If

$$\{f_j\}_{j=1}^n \subset H^\infty(D), \ H \in H^\infty(D) \ \text{and}$$

$$(2) \quad |H(z)| \leq \left( \sum_{j=1}^{n} |f_j(z)|^2 \right)^{\frac{1}{2}} \ \text{for all} \ z \in \mathbb{D}.$$  

Then

$$H^3 \in \mathcal{I} \left( \{f_j\}_{j=1}^n \right).$$
Row and Column Operators

- Given $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$, and $F(z) = (f_1(z), f_2(z), \ldots) \quad \forall z \in D$. We define the row operator

$$M_F^R : \bigoplus_1^\infty \mathcal{D} \rightarrow \mathcal{D} \text{ by }$$

$$M_F^R \left( \{h_j\}_{j=1}^{\infty} \right) = \sum_{j=1}^{\infty} f_j h_j \text{ for } \{h_j\}_{j=1}^{\infty} \in \bigoplus_1^\infty \mathcal{D}$$

- Similarly, the column operator $M_F^C : \mathcal{D} \rightarrow \bigoplus_1^\infty \mathcal{D}$ by

$$M_F^C (h) = \{f_j h\}_{j=1}^{\infty} \text{ for } \{h_j\}_{j=1}^{\infty} \in \bigoplus_1^\infty \mathcal{D}.$$
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\]
Also, it is worthwhile to note that the pointwise hypothesis

\[ F(z) F(z)^* \leq 1 \text{ for } z \in \mathbb{D}, \]

implies that the analytic Toeplitz operators \( T^R_F \) and \( T^C_F \) defined on \( \bigoplus_{1}^{\infty} H^2(\mathbb{D}) \) and \( H^2(\mathbb{D}) \) in analogy to that of \( M^R_F \) and \( M^C_F \) are bounded and

\[
\| T^R_F \| = \| T^C_F \| = \sup_{z \in \mathbb{D}} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{\frac{1}{2}} \leq 1.
\]
But, since $M(D) \not\subseteq H^\infty(D)$, the pointwise upperbound hypothesis will not be sufficient to conclude that $M^R_F$ and $M^C_F$ are bounded on Dirichlet space. However,

$$\|M^R_F\|_{B(D)} \leq \sqrt{18} \|M^C_F\|_{B(D)}, \quad \text{Trent [2002].}$$

Thus, we will replace the natural normalization that $F(z)F(z)^* \leq 1$ for all $z \in D$, by the stronger condition that $\|M^C_F\|_{B(D)} \leq 1$. 

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Thus, we will replace the natural normalization that $F(z)F(z)^* \leq 1$ for all $z \in \mathbb{D}$, by the stronger condition that $\|M_F^C\|_{B(D)} \leq 1$. 
Theorem 1.
Let $H, \{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$.
Assume

(a) $\|M^C_F\| \leq 1$ and

(b) $|H(z)| \leq \sqrt{\sum_{j=1}^{\infty} |f_j(z)|^2}$ for all $z \in \mathbb{D}$.

Then there exist $\{g_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$ with

$$\|M^C_G\| < \infty$$

and $FG^T = H^3$. 
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3. Further Discussion
Assume that $F \in \mathcal{M}_{l^2}(D)$ and $H \in \mathcal{M}(D)$ satisfy the hypotheses (a) and (b) of Theorem 1. Then we show that there exists a constant $K < \infty$, so that

$$M_{H^3} M_{H^3}^* \leq K^2 M_R^F M_R^{*F}.$$  \hspace{1cm} (4)

Given (4), a commutant lifting theorem argument completes the proof by providing a $G \in \mathcal{M}_{l^2}(D)$, so that $\|M_G^C\| \leq K$ and $F G^T = H^3$.

But (4) is equivalent to the following: there exists a constant $K < \infty$ so that, for any $h \in D$, there exists $u_h \in \bigoplus_{1}^{\infty} D$ such that

(i) $M_R^F(u_h) = H^3h$ \hspace{1cm} and

(ii) $\|u_h\|_D \leq K \|h\|_D$. 

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Outline of the Proof

Assume that \( F \in M_{l^2}(D) \) and \( H \in M(D) \) satisfy the hypotheses (a) and (b) of Theorem 1. Then we show that there exists a constant \( K < \infty \), so that

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2. \( \| u_h \|_{\mathcal{D}} \leq K \| h \|_{\mathcal{D}} \).
Lemma 1.
Let $\{c_j\}_{j=1}^{\infty} \in l^2$ and $C = (c_1, c_2, \ldots) \in B(l^2, \mathbb{C})$. Then $\exists Q$ such that entries of $Q$ are either 0 or $\pm c_j$ for some $j$ and $CC^*I - C^*C = QQ^*$. Also, Range of $Q = \text{Kernel of } C$.

We will apply this lemma in our case with $C = F(z) \forall z \in D$. It suffices to establish (i) and (ii) for any dense set of functions in $\mathbb{D}$.

For a polynomial, $h$, we take

$$u_h(z) = F(z)^* (F(z)F(z)^*)^{-1} H^3 h - Q(z) \underline{k}(z),$$

where $\underline{k}(z) \in l^2$ for $z \in \overline{\mathbb{D}}$.

We have to find $\underline{k}(z)$ so that $\overline{\partial_z} u_h = 0$ in $\mathbb{D}$. 
Sketch of the Proof

Lemma 1.
Let \( \{c_j\}_{j=1}^\infty \in l^2 \) and \( C = (c_1, c_2, \ldots) \in B(l^2, \mathbb{C}) \). Then \( \exists Q \) such that entries of \( Q \) are either 0 or \( \pm c_j \) for some \( j \) and \( CC^*I - C^*C = QQ^* \). Also, Range of \( Q = \text{Kernel of } C \).

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Lemma 1.
Let \( \{c_j\}_{j=1}^\infty \in l^2 \) and \( C = (c_1, c_2, \ldots) \in B(l^2, \mathbb{C}) \). Then \( \exists Q \) such that entries of \( Q \) are either 0 or \( \pm c_j \) for some \( j \) and \( CC^*I - C^*C = QQ^* \). Also, Range of \( Q \) = Kernel of \( C \).

We will apply this lemma in our case with \( C = F(z) \forall z \in D \).
It suffices to establish (i) and (ii) for any dense set of functions in \( \mathbb{D} \).
For a polynomial, \( h \), we take

\[
\underline{u}_h(z) = F(z)^* (F(z)F(z)^*)^{-1} H^3 h - Q(z)k(z),
\]

where \( k(z) \in l^2 \) for \( z \in \overline{\mathbb{D}} \).
We have to find \( k(z) \) so that \( \overline{\partial} z u_h = 0 \) in \( \mathbb{D} \).
Using duality, we get that

\[ u_h = \frac{F^* H^3 h}{FF^*} - Q \left( \frac{Q^* F' H^3 h}{(FF^*)^2} \right), \]

where

\[ \hat{k}(z) = -\frac{1}{\pi} \int_D \frac{k(w)}{w - z} dA(w) \text{ and } \overline{\partial} \hat{k}(z) = k(z) \text{ for } z \in \mathbb{D}. \]

It's clear that \( M^R_F(u_h) = H^3 h \) and \( u_h \) is analytic. Hence, we will be done in the smooth case if we are able to find \( K < \infty \), independent of the polynomial, such that

\[ \| u_h \|_D \leq K \| h \|_D \]
Using duality, we get that

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It’s clear that \( M^R_F(u_h) = H^3h \) and \( u_h \) is analytic. Hence, we will be done in the smooth case if we are able to find \( K < \infty \), independent of the polynomial, such that

\[ \| u_h \|_D \leq K \| h \|_D \]
Lemma 2.
Let $w$ be a harmonic function on $\overline{D}$, then
\[
\int_D \|Q'w\|_{L^2}^2 \, dA \leq 8 \|w\|_{H^2_D}^2.
\]

Lemma 3.
Let the operator $T$ be defined on $L^2(\mathbb{D}, dA)$ by
\[
(Tf)(\lambda) = \int_D \frac{f(z)}{(z - \lambda)(1 - z \overline{\lambda})} \, dA(z),
\]
for $\lambda \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA)$. Then
\[
\|Tf\|_A^2 \leq 100 \pi^2 \|f\|_A^2.
\]
Using the definition of Dirichlet norm, we have.

\[ \| u_h \|_{D}^2 = \int_{-\pi}^{\pi} \| u_h(e^{it}) \|^2 d\sigma(t) + \int_{D} \| (u_h(z))' \|^2 dA(z) \]

\[ = \int_{-\pi}^{\pi} \left\| \frac{F^*Hh}{FF^*} - Q \left( \frac{Q^*F'Hh}{(FF^*)^2} \right) \right\|^2 d\sigma(t) \]

\[ + \int_{D} \left\| \left( \frac{F^*Hh}{FF^*} - Q \left( \frac{Q^*F'Hh}{(FF^*)^2} \right) \right)' \right\|^2 dA(z) \]

Condition (b) implies that

\[ \int_{-\pi}^{\pi} \left\| \frac{F^*H^3h}{FF^*} - Q \left( \frac{Q^*F'H^3h}{(FF^*)^2} \right) \right\|^2 d\sigma(t) \leq C^2_0 \| h \|^2_{\sigma} \]
Using the definition of Dirichlet norm, we have.

\[
\|u_h\|^2_D = \int_{-\pi}^{\pi} \|u_h(e^{it})\|^2 d\sigma(t) + \int_D \| (u_h(z))' \|^2 dA(z)
\]

\[
= \int_{-\pi}^{\pi} \left\| \frac{F^*Hh}{\mathcal{F}^*} - Q \left( \frac{\overline{Q^*F'^*Hh}}{(\mathcal{F}^*)^2} \right) \right\|^2 d\sigma(t)
\]

\[
+ \int_D \left\| \left( \frac{F^*Hh}{\mathcal{F}^*} - Q \left( \frac{\overline{Q^*F'^*Hh}}{(\mathcal{F}^*)^2} \right) \right)' \right\|^2 dA(z)
\]

Condition (b) implies that

\[
\int_{-\pi}^{\pi} \left\| \frac{F^*H^3h}{\mathcal{F}^*} - Q \left( \frac{\overline{Q^*F'^*H^3h}}{(\mathcal{F}^*)^2} \right) \right\|^2 d\sigma(t) \leq C_0^2 \|h\|^2_\sigma
\]
Hence, we only need to show that

$$\int_D \left\| \left( \frac{F^* H^3 h}{FF^*} - Q \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right) \right) \right\|^2 dA(z) \leq C^2 \| h \|^2_D$$

for some $C < \infty$.

Now

$$\int_D \left\| \left( \frac{F^* H^3 h}{FF^*} - Q \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right) \right) \right\|^2 dA(z)$$

$$\leq 2 \int_D \left\| \left( \frac{F^* H^3 h}{FF^*} \right) \right\|^2 dA(z) + 2 \int_D \left\| \left( - Q \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right) \right) \right\|^2 dA(z)$$

$$\leq 4 \int_D \left\| \frac{F^* 3 H^2 H' h}{FF^*} \right\|^2 dA(z) + 8 \int_D \left\| \frac{F^* H^3 h'}{FF^*} \right\|^2 dA(z)$$

(a')

(b')

Debendra P. Banjade

Wolff's Ideal Problem in the Multiplier Algebra on Dirichlet
Wolff’s Ideal Problem in \( \mathcal{M}(D) \)

Further Discussion

Outline of the Proof

\[
\begin{align*}
(\text{c'}) & \quad + 8 \int_D \left\| \frac{F^*H^3h'F'F^*}{(FF^*)^2} \right\|^2 dA(z) \\
(\text{d'}) & \quad + 4 \int_D \left\| Q' \left( \frac{Q^*F'F^*H^3h}{(FF^*)^2} \right) \right\|^2 dA(z) \\
(\text{e'}) & \quad + 4 \int_D \left\| Q \left( \frac{Q^*F'F^*H^3h}{(FF^*)^2} \right)' \right\|^2 dA(z).
\end{align*}
\]

\[
(\text{a'}) = \int_D \left\| \frac{F^*3H^2h'}{FF^*} \right\|^2 dA(z) = 9 \int_D \left\| \frac{F^*}{\sqrt{FF^*}} \frac{H}{\sqrt{FF^*}} H H'h \right\|^2 dA(z) \leq 9 \int_D \left\| H'h \right\|^2 dA(z) \leq 36 \left\| M_H \right\|^2 \left\| h \right\|_D^2.
\]
(b') = \int_D \left\| \frac{F^*H^3h'}{FF^*} \right\|^2 dA(z) \leq \int_D \| h' \|^2 dA(z) \leq \| h \|^2_D.

(c') = \int_D \left\| \frac{F^*H^3hF'F^*}{(FF^*)^2} \right\|^2 dA(z) \leq 4 \| h \|^2_D.

(e') = 4 \int_D \left\| Q \left( \frac{Q^*F^*H^3h}{(FF^*)^2} \right)' \right\|^2 dA(z) \leq 224(14)^2 \| h \|^2_D.
So we only need to find an estimate for \((d')\).

Now,

\[
(d') = \int_D \|Q' \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right) \|^2 dA(z) = \int_D \|Q' \hat{w} \|^2 dA(z),
\]

where \(\hat{w} = \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right)\) is a smooth function on \(\overline{\mathbb{D}}\).

Therefore,

\[
\int_D \|Q' \hat{w} \|^2 dA(z) \leq 2 \int_D \|Q' \hat{w} - Q' \tilde{\hat{w}} \|^2 dA(z) + 2 \int_D \|Q' \tilde{\hat{w}} \|^2 dA(z),
\]

\(\alpha\)

where \(\tilde{\hat{w}}(z) = \int_{-\pi}^{\pi} \frac{1-|z|^2}{1-e^{-it}z} \hat{w}(e^{it}) \, d\sigma(t)\) is the harmonic extension of \(\hat{w}\) from \(\partial \mathbb{D}\) to \(\mathbb{D}\).
So we only need to find a estimate for \((d')\).

Now,

\[
(d') = \int_D \| Q' \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right) \|^2 dA(z) = \int_D \| Q' \hat{w} \|^2 dA(z),
\]

where \(\hat{w} = \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right)\) is a smooth function on \(\overline{D}\).

Therefore,

\[
\int_D \| Q' \hat{w} \|^2 dA(z) \leq 2 \int_D \| Q' \hat{w} - Q' \tilde{\hat{w}} \|^2 dA(z) + 2 \int_D \| Q' \tilde{\hat{w}} \|^2 dA(z),
\]

where \(\tilde{\hat{w}}(z) = \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \hat{w}(e^{it}) d\sigma(t)\) is the harmonic extension of \(\hat{w}\) from \(\partial D\) to \(D\).
Now, we are just left with estimating $(\alpha)$.

$$(\alpha) = \int_D \| Q' \hat{w} - Q' \tilde{\hat{w}} \|^2 dA(z)$$

$$= \int_D \| Q' \left[ \frac{1}{\pi} \int_D \frac{w(u)}{z-u} dA(u) - \int_{-\pi}^{\pi} \frac{1 - |z|^2}{1 - e^{-it}z} \hat{w}(e^{it})d\sigma(t) \right] \|^2 dA(z)$$

$$= \frac{1}{\pi^2} \int_D \| Q'(z) (1 - |z|^2) T(w)(z) \|^2 dA(z)$$

$$\leq \frac{\| M_Q \|^2}{\pi^2} \| T(w) \|^2_A$$

$$\leq 100 \pi^2 \frac{\| M_Q \|^2}{\pi^2} \| w \|^2 \text{ by Lemma 3}$$

$$\leq 100 \| M_Q \|^2 \| h \|^2_D \leq 1800 \| h \|^2_D.$$
Extension of this result to the multiplier algebra on weighted Dirichlet space with the weight $\alpha \in (0, 1)$.

Best condition for “H” itself to be in the ideal.

Extension of this result to Drury Arveson Spaces.
THANK YOU