Closed sets of approximation on non-compact Riemann surfaces

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Complex Approximation

Closed sets of Approximation

Extension Theorem

Example
Complex Approximation

- **Theorem**

  *(Runge 1885)* Suppose $K$ is compact in $\mathbb{C}$ and $f$ is analytic on $K$; further, let $\epsilon > 0$. Then there exists a rational function $R$ with poles in $\mathbb{C} \setminus K$ such that

  \[ |f(z) - R(z)| < \epsilon \quad (z \in K). \]
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$$ |f(z) - R(z)| < \epsilon \quad (z \in K). $$

In a special case where $\mathbb{C} \setminus K$ consists of the unbounded component only, for each $\epsilon > 0$, there exists a polynomial $P$ such that

$$ |f(z) - P(z)| < \epsilon \quad (z \in K). $$
One of the most important results on non-compact Riemann surfaces is the analog of Runge’s theorem on approximation by complex polynomials and rational functions, due to H. Behnke and K. Stein in 1949.
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**Theorem**

Let $K$ be a compact subset of a non-compact Riemann surface $R$, and let $R^* := R \cup \{\ast\}$ denote the one-point compactification of $R$.

Every function $f$ holomorphic on $K$ (we write $f \in \text{hol}(K)$) can be uniformly approximated on $K$ by functions holomorphic on $R$ if and only if $R^* \setminus K$ is connected.
What if we replace a compact subset $K$ by a closed subset $E$?
Closed sets of Approximation

Let \( A(E) := hol(E^\circ) \cap C(\bar{E}). \)
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- **Theorem**

  Let $E$ be a closed subset of $\mathbb{C}$.

  Then every function $f \in A(E)$ can be uniformly approximated on $E$ by entire functions if and only if $\mathbb{C}^* \setminus E$ is connected and locally connected.

Note: the same statement is true if we replace $C$ by any domain $D$ in $\mathbb{C}$ (Arakelyan, 1968)
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**Note:** the same statement is true if we replace \( \mathbb{C} \) by any domain \( D \) in \( \mathbb{C} \). (Arakelyan, 1968)
In 1972 P. M. Gauthier and W. Hengartner proved (published 1975) that the conditions in Arakelyan’s theorem are **necessary** on an arbitrary non-compact Riemann surface, but provided an example to show that, in general, they are **not sufficient**.
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The Riemann surface for this counter-example to Arakelyan’s theorem must be of infinite genus, since S. Scheinberg in 1978 and later P. M. Gauthier and W. Hengartner, proved that Arakelyan’s theorem holds (as stated) on non-compact Riemann surfaces of finite genus.
Closed sets of Approximation

Problem: Characterize the **closed** sets $E$ of a non-compact Riemann surface $R$ for which every function holomorphic on $E$ can be approximated *uniformly on $E$* by functions holomorphic (or meromorphic) on $R$. 
Closed Sets of Approximation

- The characterization of the closed sets of approximation (by holomorphic or meromorphic functions) is still an open problem, but the following cases are known:

  1) The Riemann surface $R$ is of finite genus.
  2) The set $E$ is compact.
  3) The set $E$ has no interior.
  4) The set $E$ is contained in an open set of finite genus.
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Weakly of infinite genus sets

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- Schmieder made some mistakes in one of his main theorems (“Extension Theorem”), and the procedure cannot be carried in the full generality that he claimed it could.
Schmieder’s Extension Theorem

Let $E$ be a closed subset of a non-compact Riemann surface $R$, such that there exists a neighborhood $G$ for $E$, there exists an exhaustion of $R$ by Jordan domains $D_n$, there exists Jordan domains $W_n$, with the following properties:

1) $W_n \subset \subset D_n \setminus D_{n-1}$ for any $n \in \mathbb{N}$ and $D_0 = \emptyset$
2) Each non-empty $W_n$ is of finite and positive genus.
3) $G \setminus \bigcup_{n \in \mathbb{N}} W_n$ is planar.

Let $D_n$, $W_n$ and $G$ are with the above properties, then:

a) For all $n \in \mathbb{N}$, there exists a non-compact Riemann surface $r_n$ on which $E_n := E \setminus \bigcup_{n<j} \overline{W}_j$ is relatively compact.

b) For all $n \in \mathbb{N}$, $\overline{D}_n \subset r_n$. 
Extension Theorem

Example

“The Sun” is a counter example for Extension Theorem.

The theorem and the process of the proof fail for any complement of a compact subset of $R$. 
Extension Theorem

- (N.A and A. Boivin, 2010) Extension theorem is true under the following conditions:

\[ R \setminus \bar{G} \text{ contains an unbounded Jordan arc.} \]

\[ E \text{ is a closed subset of } R \text{ and } G \text{ is an open neighbourhood of } E. \]
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- (N.A and A. Boivin, 2010) Extension theorem is true under the following conditions:

- condition: $R \setminus \bar{G}$ contains an unbounded Jordan arc. ($E$ is a closed subset of $R$ and $G$ is an open neighbourhood of $E$).

- A technical condition!
Weakly of infinite genus

- A closed subset $E$ of non-compact Riemann surface $R$ is weakly of infinite genus if:
  - It is of finite genus or,
  - The conditions of the (new) Extension Theorem are satisfied and also, very roughly speaking $E$ should not be fat at infinity and the handles should go fast to infinity.
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Weakly of infinite genus

A closed subset $E$ of a non-compact Riemann surface $R$ is said to be weakly of infinite genus-1 (wig-1), if it is possible to carry the above construction in such a way that the following inequalities are satisfied for $n = 1, 2, \ldots$

1) \[ e_n \leq \frac{\lambda_{n+2}}{2\|C_{n+1}\|_{E_n \cup D_n}} \]

2) \[ e_n \leq \frac{\sum_{j=1}^{n+2} \lambda_j}{\|2C_{n+1} + 1\|_{W_{n+1} \cap E}} \]

$\lambda_n$ is an arbitrary positive numbers satisfying $\sum \lambda_n < \infty$. 
Example

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- Let $R_0$ be an open Riemann surface with the property that

\[ R_0 \setminus K \in O_{AB} \]

whenever $K \subset R_0$ is compact and $R_0 \setminus K$ is connected.
Let $D \subset R_0$ be a parametric disc, and choose two points $p, q \in D$, $q \neq p$. Let $R = R_0 \setminus \{p\}$, $E = (R \setminus D) \cup \{q\}$ and

$$f = \begin{cases} 
0 & \text{on } R \setminus D \\
1 & \text{at the point } q.
\end{cases}$$

$R^* \setminus E$ is connected and locally connected and $f \in \text{hol}(E) \subset A(E)$. 
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Now assume that there exists $g \in \text{hol}(R)$ with $\|f - g\|_E < \varepsilon < 1/2$. It then follows that $g$ must be constant, since bounded on $R \setminus D$. This contradicts $\|f - g\|_E < 1/2$. 