Uniformity from Gromov hyperbolicity

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Abstract

We show that, in a metric space $X$ with annular convexity, uniform domains are precisely those Gromov hyperbolic domains whose quasiconformal structure on the Gromov boundary agrees with that on the metric boundary in $X$. As an application we show that quasimöbius maps between geodesic spaces with annular convexity preserve uniform domains. These results are quantitative.

1 Introduction

In this paper we study a connection between Gromov hyperbolic spaces and uniform domains. We characterize uniform domains among Gromov hyperbolic domains in metric spaces with annular convexity in terms of the quasiconformal structure of the Gromov boundary.

Let $(X, d)$ be a metric space and $C_a \geq 2$ a constant. We say $(X, d)$ is $C_a$-annular convex if for all $x \in X$, all $r > 0$ and every pair of points $y, z \in B(x, r) \setminus B(x, r/2)$ there is a path $\gamma$ joining $y$ and $z$ satisfying:

1. the length of $\gamma$ is at most $C_a d(y, z),$

2. the path $\gamma$ does not intersect the ball $B(x, r/C_a)$.

Examples of metric spaces with annular convexity include Banach spaces and Carnot groups, as well as metric measure spaces equipped with a doubling measure that supports a $(1, p)$-Poincaré inequality (see [K]).

Let $(X, d)$ be a proper metric space (that is, closed and bounded subsets are compact), and $\Omega \subset X$ a rectifiably connected open subset (every pair of points in $\Omega$ can be joined by a rectifiable path in $\Omega$) with boundary $\partial \Omega \neq \emptyset$. We say $\Omega$ is a Gromov hyperbolic domain if $\Omega$ is Gromov hyperbolic with respect to the quasihyperbolic metric $k_\Omega$ on $\Omega$. Given a bounded Gromov hyperbolic domain $\Omega$, we obtain the Gromov compactification $\bar{\Omega} = \Omega \cup \partial^* \Omega$ of $(\Omega, k_\Omega)$, where $\partial^* \Omega$ is the Gromov boundary of $\Omega$. The metric closure of $\Omega$ in $(X, d)$ is denoted $\bar{\Omega}$; since $X$ is proper and $\Omega$ is bounded, $\bar{\Omega}$ is compact. In general, the identity map $f : (\Omega, k_\Omega) \to (\Omega, d)$ does not extend to a continuous map from $\bar{\Omega}$ to $\bar{\Omega}$, and even if $f$ does extend, the extension may not be injective. However, if $\Omega$ is a uniform domain,

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then \( f \) extends to a homeomorphism from \( \overline{\Omega}^* \) to \( \overline{\Omega} \), and the restriction of the extension to the Gromov boundary is a quasimöbius map with respect to the visual metric on \( \partial^* \Omega \) (see [BHK]). The main result (Theorem 9.1) of this paper is that uniform domains are the only Gromov hyperbolic domains in an annular convex proper metric space with the above property. As an application of the main result we demonstrate that quasimöbius maps preserve uniform domains (Theorem 10.1): if \( \Omega \) is a domain in an annular convex proper metric space and \( \Omega \) is quasimöbius equivalent to a uniform domain in some metric space, then \( \Omega \) is also uniform.

In the setting of Euclidean spaces and spheres Theorem 9.1 has been proven by Bonk, Heinonen, and Koskela [BHK], and Väisälä [V1] proved this theorem for domains in Banach spaces. The proof in [BHK] makes use of the notion of moduli of curve families, and therefore does not extend to metric spaces that have no “nice” measure. The proof in [V1] uses only metric properties. Our proof follows the general outline of the arguments found there, but it contains the following two new ingredients.

In [V1] the theorem was first proved for unbounded domains in Banach spaces, and then inversions in Banach spaces were used to reduce the study of bounded domains to the study of unbounded domains. To follow this strategy, we use a notion of “inversion” in general metric spaces, see Section 4 or [BHX] for more details.

The second crucial property used in [V1] is the fact that spheres in Banach spaces are 2-quasiconvex. Our replacement for this property is the annular convexity property of the underlying metric space \( X \). Points in \( \Omega \) can be classified as annulus points or arc points (see Section 7). A consequence of the annular convexity property is that each arc point lies on an anchor. This was first shown in [BHK] in the Euclidean setting. Under the assumption of annular convexity we establish a slightly weaker version of these facts sufficient for the proof of Theorem 9.1. It is not clear whether Theorem 9.1 still holds if the metric space is not annular convex.

As in [V1] we interpret the cross ratio in the Gromov boundary with respect to a visual metric \( d_{y,\epsilon} \) in terms of distances between certain geodesics (see Section 5). Let \((Y,d)\) be a proper geodesic Gromov hyperbolic space, \( Q = (\xi_1, \xi_2, \xi_3, \xi_4) \) a quadruple of distinct points in the Gromov boundary \( \partial^* Y \). Fix any geodesic \([\xi_i, \xi_j]\) \((1 \leq i, j \leq 4)\) joining \( \xi_i \) and \( \xi_j \). The cross ratio of \( Q \) with respect to \( d_{y,\epsilon} \), denoted \( \text{cr}(Q, d_{y,\epsilon}) \), satisfies

\[
\text{cr}(Q, d_{y,\epsilon}) \approx \begin{cases} 
eq (|\xi_1, \xi_4|,|\xi_2, \xi_3|) & \text{if } d([\xi_1, \xi_4], [\xi_2, \xi_3]) \geq d([\xi_1, \xi_3], [\xi_2, \xi_4]), \\ e^{-\epsilon d([\xi_1, \xi_3], [\xi_2, \xi_4])} & \text{otherwise.} \end{cases}
\]

This interpretation of cross ratio is quite convenient in studying the quasiconformal structure of the Gromov boundary, and allows us to simplify some of the arguments found in [V1].

**Notation:** In this note \((X,d)\) denotes a metric space, \( B(x, r) = \{ y \in X : d(x, y) < r \} \) is the open ball with center \( x \in X \) and radius \( r > 0 \), and \( S(x, r) = \{ y \in X : d(y, x) = r \} \) is the sphere of radius \( r \) and center \( x \). The image of a path \( \alpha : [a, b] \to X \) is denoted \( |\alpha| \), and \( \ell_d(\alpha) \) denotes the \( d \)-length of \( \alpha \) (we simply use \( \ell(\alpha) \) if the metric \( d \) in question is clear). We use \( \alpha : x \leadsto y \) to indicate a path \( \alpha : [a, b] \to X \) with \( \alpha(a) = x \) and \( \alpha(b) = y \). If a path \( \alpha : [a, b] \to X \) is an embedding and \( x, y \in |\alpha| \), then both \( \alpha_{xy} \) and \( \alpha[x, y] \) denote the subpath of \( \alpha \) from \( x \) to \( y \); we prefer the second notation when a path already has a subscript, say,
If $A \subset X$ and $r > 0$, then $N_r(A) = \{y \in X : d(y, x) \leq r \text{ for some } x \in A\}$ denotes the closed $r$-neighborhood of $A$. For two bounded subsets $A, B \subset X$,

$$HD_d(A, B) = \inf\{r > 0 : B \subset N_r(A) \text{ and } A \subset N_r(B)\}$$

denotes the Hausdorff distance between $A$ and $B$; if the metric $d$ in question is clear, we use $HD(A, B)$. Given two real numbers $a, b$, we denote the smaller of these by $a \wedge b$. By $c = c(\delta, \eta, C_a)$ we mean $c$ depends only on $\delta, \eta$ and $C_a$. We say that a metric space is a geodesic space if every pair of points $x, y$ in that space can be joined by a geodesic, that is, a path with length $d(x, y)$. A metric space is said to be $c$-quasiconvex for some $c \geq 1$ if each pair of points $x, y$ in the space can be joined by a curve of length no more than $cd(x, y)$. Thus, a geodesic space is 1-quasiconvex. It should be noted that if $\Omega$ is a non-empty open subset of a $c$-quasiconvex space, then for every $x \in \Omega$ there exists $r_x > 0$ such that all $y, z \in B(x, r_x)$ can be joined by a curve in $\Omega$ with length at most $cd(y, z)$.

### 2 Quasihyperbolic metrics and uniform domains

In this section we recall some basic facts about quasihyperbolic metrics. While we do not give proofs for most of these facts, we do provide citations the reader can refer to for them.

Let $\Omega$ be an open subset of a metric space $(X, d)$. We say $\Omega$ is rectifiably connected if each pair of points $x, y \in \Omega$ can be joined by a rectifiable path $\gamma \subset \Omega$. The boundary $\partial \Omega$ of $\Omega$ is the set $\Omega \setminus \Omega$.

Let $(X, d)$ be a proper metric space, and $\Omega \subset X$ an open rectifiably connected subset of $X$ with $\partial \Omega \neq \emptyset$. For $x \in \Omega$, we denote $\delta_\Omega(x) = d(x, X \setminus \Omega)$. The quasihyperbolic metric $k_\Omega$ on $\Omega$ is defined as follows: for $x, y \in \Omega$,

$$k_\Omega(x, y) := \inf \int_{\gamma} ds(z),$$

where the infimum is over all rectifiable curves $\gamma$ in $\Omega$ joining $x$ and $y$, and $ds$ denotes the arc length element along $\gamma$. It can be shown that $k_\Omega$ indeed is a metric.

The length metric $l_\Omega$ on $\Omega$ is given by $l_\Omega(x, y) = \inf_{\gamma} \ell_d(\gamma)$ for $x, y \in \Omega$, where the infimum is over paths in $\Omega$ joining $x$ and $y$.

**Proposition 2.1** (Proposition 2.8 of [BHK]). *If the identity map $f : (\Omega, d) \to (\Omega, l_\Omega)$ is a homeomorphism, then $f^{-1} : (\Omega, d) \to (\Omega, k_\Omega)$ is also a homeomorphism with $(\Omega, k_\Omega)$ a proper geodesic space.*

It should be noted that if $X$ is a $c$-quasiconvex space and $\Omega$ is an open connected set (that is, a domain), then id : $(\Omega, d) \to (\Omega, l_\Omega)$ is indeed a homeomorphism. Thus, in this paper the assumption in Proposition 2.1 is always satisfied and hence the space $(\Omega, k_\Omega)$ is a proper geodesic space.

**Lemma 2.2.** *If $x, y \in \Omega$ and $\alpha : x \sim y$ is a rectifiable arc in $\Omega$, then

$$\ell_d(\alpha) \leq (e^{k_\Omega(\alpha)} - 1) \delta_\Omega(x).$$*
Proof. Without loss of generality, let \( \alpha \) be parametrized by \( d \)-arclength. Then

\[
\ell_{k_{\Omega}}(\alpha) = \int_0^{\ell_d(\alpha)} \frac{1}{\delta_{\Omega}(\alpha(t))} \, dt \geq \int_0^{\ell_d(\alpha)} \frac{1}{\delta_{\Omega}(x) + t} \, dt = \log \left( 1 + \frac{\ell_d(\alpha)}{\delta_{\Omega}(x)} \right).
\]

Let \( x, y \in \Omega \) and \( \alpha \) be a quasihyperbolic geodesic from \( x \) to \( y \). Then \( \ell_{k_{\Omega}}(\alpha) = k_{\Omega}(x, y) \) and \( \ell_d(\alpha) \geq d(x, y) \). Since we may assume \( \delta_{\Omega}(x) \leq \delta_{\Omega}(y) \), the proof of Lemma 2.2 shows that the following holds for all \( x, y \in \Omega \):

\[
k_{\Omega}(x, y) \geq \log \left( 1 + \frac{d(x, y)}{\delta_{\Omega}(x) \wedge \delta_{\Omega}(y)} \right).
\]

Since \( \delta_{\Omega}(y) \leq \delta_{\Omega}(x) + d(x, y) \), we have

\[
k_{\Omega}(x, y) \geq \left| \log \frac{\delta_{\Omega}(y)}{\delta_{\Omega}(x)} \right|.
\]

(1)

**Lemma 2.3 (Lemma 2.13 of [BHK]).** Let \((X, d)\) be a proper metric space and \( \Omega \subset X \) a rectifiably connected open subset with \( \partial \Omega \neq \emptyset \). If \( \gamma : [0, 1] \rightarrow \Omega \) is a curve that satisfies

\[
\min\{\ell_d(\gamma|_{[0,t]}), \ell_d(\gamma|_{[t,1]})\} \leq A \delta_{\Omega}(\gamma(t))
\]

for all \( t \in [0, 1] \), then with \( x = \gamma(0) \) and \( y = \gamma(1) \),

\[
\ell_{k_{\Omega}}(\gamma) \leq 4A \log \left( 1 + \frac{\ell_d(\gamma)}{\delta_{\Omega}(x) \wedge \delta_{\Omega}(y)} \right).
\]

The following is a modification to our setting of Lemma 3.5 of [V1]. Since we replace the 2-quasiconvexity of spheres with the annular convexity property, and hence the estimates we obtain here are necessarily weaker than those in [V1].

**Lemma 2.4.** Suppose \((X, d)\) is \( C_a \)-annular convex for some \( C_a \geq 2 \). Let \( \alpha : x \sim y \) be a quasihyperbolic geodesic in \( \Omega \), \( a \in \partial \Omega \), and \( t > 0 \).

(i) If \( B(a, 16C_a^2 t) \setminus B(a, e^{-4C_a^3 t}/2) \subset \Omega \) and \( x, y \in \Omega \setminus B(a, 8C_a t) \), then \( |\alpha| \subset \Omega \setminus B(a, e^{-4C_a^3 t}) \).

(ii) If \( B(a, 8C_a t) \setminus B(a, t/C_a) \subset \Omega \) and \( x, y \in \Omega \cap B(a, 4t) \), then \( |\alpha| \subset \Omega \cap B(a, 8e^{4C_a^3 t}) \).

Proof. We first prove (i). Suppose that \( |\alpha| \cap B(a, e^{-4C_a^3 t}) \neq \emptyset \). Then \( \alpha \) must intersect \( B(a, 8C_a t) \) as well, and so we can choose \( z_1, z_2 \in |\alpha| \cap S(a, 8C_a t) \) such that the subcurve \( \alpha_{z_1z_2} \) of \( \alpha \) satisfies both \( |\alpha_{z_1z_2}| \subset \overline{B(a, 8C_a t)} \) and \( |\alpha_{z_1z_2}| \cap B(a, e^{-4C_a^3 t}) \neq \emptyset \).

As \( X \) is annular convex, there is a path \( \gamma : z_1 \rightsquigarrow z_2 \) with \( \ell_d(\gamma) \leq C_a d(z_1, z_2) \leq C_a 2(8C_a t) = 16C_a^2 t \) and \( |\gamma| \subset B(a, 8C_a^2 t) \setminus B(a, 8t) \subset \Omega \). Hence by the hypothesis of (i), for every \( w \in |\gamma| \) we have

\[
\delta_{\Omega}(w) \geq \min\{8C_a^2 t, 8t - e^{-4C_a^3 t}/2\} = 8t - e^{-4C_a^3 t}/2 \geq 4t.
\]
Thus the curve
\[
\ell_{k\Omega}(\gamma) = \int_{\gamma} \frac{1}{\delta_O(w)}|dw| \leq \frac{1}{4t} \ell_d(\gamma) \leq \frac{1}{4t} 16C^2_a t = 4C^2_a,
\]
hence we see that \(k_{\Omega}(z_1, z_2) \leq 4C^2_a\). Since \(\alpha_{z_1z_2}\) is a quasihyperbolic geodesic, we have
\[
\ell_{k\Omega}(\alpha_{z_1z_2}) = k_{\Omega}(z_1, z_2) \leq 4C^2_a. \tag{2}
\]
By assumption, there is a point \(z \in |\alpha_{z_1z_2}| \cap B(a, e^{-4C^3_a} t)\). By Lemma 2.2,
\[
\ell_{k\Omega}(\alpha_{z_1z_2}) = \ell_{k\Omega}(\alpha_{z_1z}) + \ell_{k\Omega}(\alpha_{z_2z}) \geq \log \left[ \left(1 + \frac{\ell_d(\alpha_{z_1z})}{\delta_O(z)} \right) \left(1 + \frac{\ell_d(\alpha_{z_2z})}{\delta_O(z)} \right) \right]
\geq \log \left(1 + \frac{\ell_d(\alpha_{z_2z})}{\delta_O(z)} \right).
\]
However, as \(\ell_d(\alpha_{z_2z}) \geq 8C_a t - e^{-4C^3_a} t\) and \(\delta_O(z) \leq e^{-4C^3_a} t\), we see that
\[
\ell_{k\Omega}(\alpha_{z_1z_2}) \geq \log \left(1 + \frac{8C_a t - e^{-4C^3_a} t}{e^{-4C^3_a} t} \right) = \log \left( \frac{8C_a t e^{4C^3_a}}{e^{4C^3_a}} \right) \geq 4C^3_a. \tag{3}
\]
Combining inequalities (2) and (3) we obtain \(4C^2_a \geq 4C^3_a\), a contradiction because \(C_a \geq 2\). Thus the curve \(\alpha\) cannot intersect the ball \(B(a, e^{-4C^3_a} t)\).

Now we prove (ii). To do so, suppose that \(|\alpha| \cap S(a, 8e^{4C^3_a} t) \neq \emptyset\). Then clearly \(|\alpha|\) intersects the sphere \(S(a, 4t)\), and so there are points \(w_1, w_2 \in |\alpha| \cap S(a, 4t)\) satisfying \(|\alpha_{w_1w_2}| \cap S(a, 8e^{4C^3_a} t) \neq \emptyset\) and \(|\alpha_{w_1w_2}| \cap B(a, 4t) = \emptyset\).

By the annular convexity of \(X\), there is a rectifiable curve \(\gamma\) joining \(w_1\) and \(w_2\) in the annulus \(B(a, 4C_a t) \setminus B(a, 4t/C_a) \subset \Omega\) with \(\ell_d(\gamma) \leq C_a d(w_1, w_2) \leq 8C_a t\). For every \(z \in |\gamma|\),
\[
\delta_O(z) \geq \min \left\{ 8C_a t - 4C_a t, \frac{4t}{C_a} - \frac{t}{C_a} \right\} = \frac{3t}{C_a}.
\]
Therefore,
\[
k_O(w_1, w_2) \leq \ell_{k\Omega}(\gamma) \leq \frac{C_a}{3t} \ell_d(\gamma) \leq \frac{C_a}{3t} 8C_a t = \frac{8}{3} C^2_a.
\]
Since \(\alpha\) is a quasihyperbolic geodesic, we see that
\[
\ell_{k\Omega}(\alpha_{w_1w_2}) = k_{\Omega}(w_1, w_2) \leq \frac{8}{3} C^2_a. \tag{4}
\]
Meanwhile, by Lemma 2.2 and by the facts that \(\ell_d(\alpha_{w_1w_2}) \geq 8e^{4C^3_a} t - 4t \geq 4e^{4C^3_a} t\) and \(\delta_O(w_1) \leq 4t\),
\[
\ell_{k\Omega}(\alpha_{w_1w_2}) \geq \ell_d \left( 1 + \frac{\ell_d(\alpha_{w_1w_2})}{\delta_O(w_1)} \right) \geq \log \left(1 + \frac{4e^{4C^3_a} t}{4t} \right) \geq \log \left( e^{4C^3_a} \right) = 4C^3_a.
\]
By inequality (4), we now get the contradiction \(4C^3_a \leq \frac{8}{3} C^2_a\) as \(C_a \geq 2\). This concludes the proof of this lemma. \(\square\)
Given \( c \geq 1 \), a path \( \gamma : [0,1] \rightarrow \Omega \) is called a \( c \)-uniform curve if

1. \( \ell_d(\gamma) \leq c d(\gamma(0), \gamma(1)) \);
2. \( c \delta_\Omega(z) \geq \min\{\ell_d(\gamma|[0,t]), \ell_d(\gamma|[t,1])\} \) for all \( t \in [0,1] \).

An open subset \( \Omega \subset X \) with \( \partial \Omega \neq \emptyset \) is called a \( c \)-uniform domain for some \( c \geq 1 \) if every two points \( x,y \in \Omega \) can be joined by a \( c \)-uniform curve. If \( \Omega \) is equipped with more than one metric, then to specify the metric with respect to which \( \Omega \) is uniform we say that \((\Omega, d)\) is a uniform domain.

**Lemma 2.5.** Let \( x_1 \in \Omega \) and \( x_2 \in \overline{\Omega} \) such that \( \delta_\Omega(x_1) \geq d(x_1, x_2) \), and \( \gamma \) a geodesic in \( X \) with respect to the metric \( d \) connecting \( x_1 \) and \( x_2 \). Then \( \gamma \{x_2\} \subset \Omega \), \( \gamma \) is a \( 1 \)-uniform curve, and furthermore, \( \delta_\Omega(x) \geq \ell_d(\gamma_{x_2}) = d(x_1, x_2) \) for all \( x \in |\gamma| \).

**Proof.** By assumption, \( x_2 \in \overline{B}(x_1, \delta_\Omega(x_1)) \cap \overline{\Omega} \). Let \( \gamma : [0,d(x_1, x_2)] \rightarrow X \) be the arc-length parametrization of \( \gamma \) with \( \gamma(0) = x_2 \), and \( \gamma(d(x_1, x_2)) = x_1 \). Then, for every \( z \in |\gamma| \{x_2\} \) we have \( d(x_1, z) < d(x_1, x_2) \), and hence \( z \in B(x_1, \delta_\Omega(x_1)) \subset \Omega \).

Now, for \( t \in (0, d(x_1, x_2)) \), we have \( d(\gamma(t), x_1) = d(x_1, x_2) - t \), whence,

\[
\delta_\Omega(\gamma(t)) \geq \delta_\Omega(x_1) - d(x_1, x_2) + t = t + [\delta_\Omega(x_1) - d(x_1, x_2)] \geq t.
\]

\( \square \)

**Proposition 2.6.** Let \( X \) be a proper geodesic metric space and \( \Omega \subset X \) an open subset with \( \partial \Omega \neq \emptyset \). Suppose \( x_0 \in \Omega \). Then there exists a point \( b \in \partial \Omega \) and a curve \( \gamma : x_0 \prec b \) such that

1. \( \ell_d(\gamma) = \delta_\Omega(x_0) \),
2. \( |\gamma| \subset \overline{B}(x_0, \delta_\Omega(x_0)) \),
3. \( |\gamma| \{b\} \subset \Omega \),
4. for every \( x \in |\gamma| \{b\} \) we have \( \ell_d(\gamma_{0x}) = \delta_\Omega(x) \),
5. \( \gamma \) is a geodesic with respect to the metric \( d \),
6. \( \gamma \) is a quasihyperbolic geodesic in \( \Omega \).

**Proof.** Since \( X \) is proper, we can choose \( b \in \partial \Omega \) such that \( \delta_\Omega(x_0) = d(x_0, b) \), and we set \( \gamma \) to be a geodesic in \( X \) with respect to the metric \( d \) joining \( x_0 \) and \( b \). Then by Lemma 2.5 we see that the conditions (i) through (v) are satisfied by \( \gamma \). To see (vi), let \( \gamma : [0, \delta_\Omega(x_0)] \rightarrow X \) be the arclength parametrization of \( \gamma \) with respect to the metric \( d \), with \( \gamma(0) = b \) and \( \gamma(\delta_\Omega(x_0)) = x_0 \). Let \( 0 < t_1 < t_2 \leq \delta_\Omega(x_0) \). Since \( \delta_\Omega(\gamma(t)) = t \), inequality (1) implies that \( k_\Omega(\gamma(t_1), \gamma(t_2)) \geq \log(t_2/t_1) \). On the other hand,

\[
k_\Omega(\gamma(t_1), \gamma(t_2)) \leq \ell_{k_\Omega}(\gamma|_{t_1, t_2}) = \int_{t_1}^{t_2} \frac{1}{\delta_\Omega(\gamma(t))} dt = \int_{t_1}^{t_2} \frac{1}{t} dt = \log(\frac{t_2}{t_1}).
\]

Hence \( \gamma \) is a quasihyperbolic geodesic in \( \Omega \). \( \square \)
Given two rectifiable curves $\alpha, \beta$ in a metric space $(Y, d)$, we say a map $f : |\alpha| \to |\beta|$ is a length map with respect to the metric $d$ if for all $x, y \in |\alpha|$ we have $\ell_d(\beta f(x), f(y)) = \ell_d(\alpha_x y)$.

**Lemma 2.7 (Lemma 3.3 of [V1]).** If $\alpha$ and $\beta$ are curves in $(\Omega, k_{\Omega})$ with $\ell_{k_{\Omega}}(\alpha) \leq \ell_{k_{\Omega}}(\beta)$, and $f : |\alpha| \to |\beta|$ is a length map (with respect to $k_{\Omega}$) with $k_{\Omega}(f(x), x) \leq c$ whenever $x \in |\alpha|$, then

$$e^{-c} \ell_d(\alpha) \leq \ell_d(f(|\alpha|)) \leq e^c \ell_d(\alpha).$$

## 3 Gromov hyperbolic spaces

In this section we review some basic facts about Gromov hyperbolic spaces. See [BHK], [CDP], [GdlH], [V1], and references therein for more details.

Let $(Y, d)$ be a proper geodesic space and $\delta \geq 0$ a constant. We say $(Y, d)$ is $\delta$-hyperbolic if geodesic triangles in $Y$ are $\delta$-thin, that is, for any $x, y, z \in Y$, any geodesics $\gamma_1 : x \sim y$, $\gamma_2 : y \sim z$, $\gamma_3 : z \sim x$, we have $|\gamma_3| \leq N_\delta(|\gamma_1| \cup |\gamma_2|)$. A space $(Y, d)$ is Gromov hyperbolic if it is $\delta$-hyperbolic for some $\delta \geq 0$. Let $w \in Y$ be a (fixed) base point. The Gromov product of $x, y \in Y$ based at $w$ is:

$$(x|y)_w = \frac{1}{2} (d(x, w) + d(y, w) - d(x, y)).$$

If $Y$ is $\delta$-hyperbolic, a sequence of points $\{y_i\}$ is said to goes to infinity if $\lim_{i,j \to \infty} (y_i|y_j)_w = \infty$ for some (or any) base point $w \in Y$. Two sequences $\{x_i\}$ and $\{y_i\}$ going to infinity are equivalent if $\lim_{i,j \to \infty} (x_i|y_j)_w = \infty$. The Gromov boundary $\partial^* Y$ of $Y$ is the set of equivalence classes of sequences going to infinity. If the metric $d$ in question is clear, we simply denote the Gromov boundary by $\partial Y$, and set $\overline{Y}^* = Y \cup \partial^* Y$. If $\xi \in \partial^* Y$ and a sequence of points $\{x_i\}$ represent $\xi$ we write $\{x_i\} \to \xi$. Interested reader may refer to Chapter 7 of [GdlH] for more details.

If $\gamma : [0, \infty) \to Y$ is a geodesic (ray), then one sees easily from the definition that $\{\gamma(t)\}$ goes to infinity as $t \to \infty$ and hence represent some $\xi \in \partial^* Y$. In this case we say $\gamma(0)$ and $\xi$ are the endpoints of $\gamma$. Similarly, for any complete geodesic $\gamma : \mathbb{R} \to Y$ there are $\xi_+, \xi_- \in \partial^* Y$ such that $\{\gamma(t)\} \to \xi_+$ as $t \to \infty$ and $\{\gamma(t)\} \to \xi_-$ as $t \to -\infty$. We say $\xi_+$ and $\xi_-$ are the endpoints of $\gamma$. A proper geodesic $\delta$-hyperbolic space has the visibility property: given any two distinct points $a, b \in \overline{Y}^*$, there is a geodesic $\gamma$ with $a$ and $b$ as endpoints (Proposition 2.1 in Chapter 2 of [CDP]). Geodesic triangles in $\overline{Y}^*$ are $24\delta$-thin (Proposition 2.2 in Chapter 2 of [CDP]). From this it can be seen that if $\alpha, \beta$, and $\gamma$ are the three sides of a geodesic triangle, then there is some $x \in \gamma$ satisfying $d(x, |\alpha|) \leq 24\delta$ and $d(x, |\beta|) \leq 24\delta$.

Let $w \in Y$ be a base point. The Gromov product of two points $\xi, \eta \in \partial^* Y$ is defined as follows:

$$(\xi|\eta)_w = \sup_{i,j \to \infty} \liminf (x_i|y_j)_w,$$

where the supremum is taken over all sequences $\{x_i\} \to \xi$, $\{y_i\} \to \eta$. One can show that $(\xi|\eta)_w - 2\delta \leq \liminf_{i,j \to \infty} (x_i|y_j)_w \leq (\xi|\eta)_w$ for all $w \in Y$, all $\xi, \eta \in \partial^* Y$ and all sequences
\{x_i\} \to \xi, \{y_i\} \to \eta; see Chapter 7 of [GdlH]. Similarly, the Gromov product of \( x \in Y \) and \( \eta \in \partial^* Y \) is defined to be

\[(x|\eta)_w = \sup_{i \to \infty} \lim_{i \to \infty} (x|y_i)_w,\]

where the supremum is taken over all sequences \( \{y_i\} \to \eta \).

We define a topology on \( \overline{Y}^* \) by specifying when a sequence of points \( x_i \in \overline{Y}^* \) converges to a point \( \xi \in \overline{Y}^* \): if \( \xi \in Y \), then \( x_i \to \xi \) means \( d(\xi, x_i) \to 0 \) as \( i \to \infty \); and if \( \xi \in \partial^* Y \), then \( x_i \to \xi \) means \( (\xi|x_i)_w \to \infty \) for some (equivalently all) \( w \in Y \) as \( i \to \infty \). In this topology, \( \overline{Y}^* \) is compact and \( Y \) is a dense open subset. The induced topology on \( Y \) agrees with the metric topology on \( Y \).

Given \( \epsilon > 0, w \in Y \) and \( \xi, \eta \in \partial^* Y \), let

\[\rho_{w,\epsilon}(\xi, \eta) = e^{-\epsilon(\xi|\eta)_w}.\]

**Proposition 3.1** (**Proposition 10 of [GdlH], Chapter 7**). Let \( \epsilon_0(\delta) = \min\{1, \frac{\delta}{M}\} \). Then for any \( \delta \)-hyperbolic metric space \( Y \), any base point \( w \in Y \), and any \( 0 < \epsilon \leq \epsilon_0 \), there is a metric \( d_{w,\epsilon} \) on \( \partial^* Y \) such that for all \( \xi, \eta \in \partial^* Y \),

\[\frac{1}{2} d_{w,\epsilon}(\xi, \eta) \leq \rho_{w,\epsilon}(\xi, \eta) \leq d_{w,\epsilon}(\xi, \eta).

A metric \( d_{w,\epsilon} \) satisfying the conclusion of Proposition 3.1 is called a *visual metric*.

**Definition 3.2.** Let \( L \geq 1 \) and \( A \geq 0 \). A (not necessarily continuous) map \( \gamma : I \to Y \) on an interval \( I \) is an \((L, A)\)-*quasigeodesic* if for all \( t_1, t_2 \in I \) we have

\[L^{-1}|t_2 - t_1| - A \leq d(\gamma(t_1), \gamma(t_2)) \leq L|t_2 - t_1| + A.\]

Note that an \((L, A)\)-quasigeodesic is a geodesic if and only if \( L = 1 \) and \( A = 0 \). An important property of Gromov hyperbolic spaces is the stability of quasigeodesics. It says that quasigeodesics are close to geodesics (see also [V3]):

**Lemma 3.3** (**Theorem 1.2 and Theorem 3.1 of [CDP], Chapter 3**). Given any \( \delta \geq 0, L \geq 1, \) and \( A \geq 0 \), there is a constant \( M = M(\delta, L, A) \) such that whenever \( Y \) is a proper geodesic \( \delta \)-hyperbolic space, the following conditions hold:

1. If \( \alpha : [a, b] \to Y \) and \( \alpha' : [a', b'] \to Y \) are two \((L, A)\)-quasigeodesics with \( \alpha(a) = \alpha'(a') \) and \( \alpha(b) = \alpha'(b') \), then the Hausdorff distance \( HD(|\alpha|, |\alpha'|) \leq M;\)

2. If \( \alpha : \mathbb{R} \to Y \) is an \((L, A)\)-quasigeodesic, then there exists a geodesic \( \alpha' : \mathbb{R} \to Y \) such that \( HD(|\alpha|, |\alpha'|) \leq M.\)

Lemma 3.3 (2) implies that every quasigeodesic \( \alpha : \mathbb{R} \to Y \) has two endpoints \( \xi_+, \xi_- \) in \( \partial^* Y \). Since two complete geodesics with the same endpoints in a \( \delta \)-hyperbolic space has Hausdorff distance at most \( 2\delta \) from each other, by replacing \( M \) with \( 2\delta + 2M \), we have that \( HD(|\alpha|, |\alpha'|) \leq M \) for any two \((L, A)\)-quasigeodesics with the same endpoints.

We also recall the following two results.
Theorem 3.4 (Chapter 8 of [CDP]). Let \((Y, d)\) be a \(\delta\)-hyperbolic space, \(y_0 \in Y\), and \(Y_0 = \{y_0, y_1, \ldots, y_n\}\) be a set of \(n + 1\) points in \(Y \cup \partial Y\). Let \(X\) denote the union of geodesics \([y_0, y_i]\) connecting \(y_0\) and \(y_i\), \(1 \leq i \leq n\), and we choose a positive integer \(k\) such that \(2n \leq 2^k + 1\). Then there exists a simplicial tree, denoted \(T(X)\), and a continuous map \(u : X \to T(X)\) which satisfies the following properties:

1. For each \(i\), the restriction of \(u\) to the geodesic \([y_0, y_i]\) is an isometry;
2. For every \(x, y \in X\) we have \(d(x, y) - 2k\delta \leq d(u(x), u(y)) \leq d(x, y)\).

Lemma 3.5 (Lemma 2.17 of [V3]). Suppose \((Y, d)\) is \(\delta\)-hyperbolic and \(\alpha_1 : a_1 \rightsquigarrow b_1\), \(\alpha_2 : a_2 \rightsquigarrow b_2\) are two geodesics with \(\ell(\alpha_1) \leq \ell(\alpha_2)\). If \(d(a_1, a_2) \leq \mu, d(b_1, |\alpha_2|) \leq \mu\) for some \(\mu \geq 0\), and \(f : |\alpha_1| \to |\alpha_2|\) is the length map with \(f(a_1) = f(a_2)\), then \(d(f(x), x) \leq 8\delta + 5\mu\) for all \(x \in |\alpha_1|\).

4 Inversions in metric spaces

In this section we recall the notion of inversions in metric spaces and collect related facts useful in this paper. See [BHX] for more details.

Let \((X, d)\) be a metric space and \(Q = (x_1, x_2, x_3, x_4)\) a quadruple of distinct points in \(X\). The cross ratio of \(Q\) with respect to \(d\) is the number

\[
\text{cr}(Q, d) = \frac{d(x_1, x_3) d(x_2, x_4)}{d(x_1, x_4) d(x_2, x_3)}.
\]

Let \(\eta : [0, \infty) \to [0, \infty)\) be a homeomorphism. A homeomorphism \(f : (X, d_1) \to (Y, d_2)\) between two metric spaces is called an \(\eta\)-quasimöbius map if for each quadruple of distinct points \(Q = (x_1, x_2, x_3, x_4)\) in \(X\),

\[
\text{cr}(f(Q), d_2) \leq \eta(\text{cr}(Q, d_1)),
\]

where \(f(Q) = (f(x_1), f(x_2), f(x_3), f(x_4))\). We say a homeomorphism \(f : (X, d_1) \to (Y, d_2)\) is quasimöbius if it is \(\eta\)-quasimöbius map for some \(\eta\). A homeomorphism \(f : (X, d_1) \to (Y, d_2)\) between two metric spaces is called an \(\eta\)-quasisymmetric map if for all triples of distinct points \((x_1, x_2, x_3)\) in \(X\),

\[
\frac{d_2(f(x_1), f(x_2))}{d_2(f(x_1), f(x_3))} \leq \eta\left(\frac{d_1(x_1, x_2)}{d_1(x_1, x_3)}\right).
\]

We say a homeomorphism \(f : (X, d_1) \to (Y, d_2)\) is quasisymmetric if it is \(\eta\)-quasisymmetric for some \(\eta\). A quasisymmetric homeomorphism is quasimöbius, but a quasimöbius homeomorphism may not be quasisymmetric. However, a quasimöbius homeomorphism between bounded metric spaces is quasisymmetric.

Let \((X, d)\) be a proper metric space and \(p \in X\). Set \(I_p(X) = X \setminus \{p\}\) if \(X\) is bounded and \(I_p(X) = (X \setminus \{p\}) \cup \{\infty\}\) if \(X\) is unbounded, where \(\infty\) is a point not in \(X\). We now define a metric \(d_p\) on \(I_p(X)\).
Consider the function \( f_p : I_p(X) \times I_p(X) \to [0, \infty) \) given by

\[
f_p(x, y) = \begin{cases} 
\frac{d(x, y)}{d(x, p) d(y, p)} & \text{if } x, y \in X \setminus \{p\}, \\
\frac{1}{d(x, p)} & \text{if } y = \infty \text{ and } x \in X \setminus \{p\}, \\
0 & \text{if } x = \infty = y.
\end{cases}
\]

For \( x, y \in I_p(X) \), we define

\[
d_p(x, y) := \inf \sum_{i=0}^{k-1} f_p(x_i, x_{i+1}),
\]

where the infimum is taken over all finite sequences of points \( x_0, \ldots, x_k \in I_p(X) \) with \( x_0 = x \) and \( x_k = y \).

**Theorem 4.1 ([BHX]).** The following holds for all \( x, y \in I_p(X) \):

\[
\frac{1}{4} f_p(x, y) \leq d_p(x, y) \leq f_p(x, y).
\]

In particular, \( d_p \) is a metric on \( I_p(X) \) and the identity map \( f : (X \setminus \{p\}, d) \to (X \setminus \{p\}, d_p) \) is an \( \eta \)-quasimöbius homeomorphism with \( \eta(t) = 16t \). Furthermore:

1. If \( (X, d) \) is \( c \)-quasiconvex and \( c \)-annular convex, then \((I_p(X), d_p)\) is \( c' \)-quasiconvex and \( c' \)-annular convex with \( c' \) depending only on \( c \);

2. Let \( \Omega \subset (X, d) \) be a \( c \)-quasiconvex open subset of \( X \) with at least two boundary points, and \( p \in \partial \Omega \). Denote by \( k_{\Omega, p} \) the quasihyperbolic metric on \( \Omega \) induced by the metric \( d_p \). If \( d_0 := \text{diam}(\Omega, d) < \infty \) and \( d'_0 := \text{diam}(\partial \Omega, d) > 0 \), then the restriction of the identity map \( f|_\Omega : (\Omega, k_{\Omega}) \to (\Omega, k_{\Omega, p}) \) is \( M \)-bilipschitz with \( M = \max \{200c, 16c d_0/d'_0\} \);

3. Let \( \Omega \subset X \) be an open subset with \( p \in \partial \Omega \). If \( (\Omega, d_p) \) is \( c_1 \)-uniform and \( (X, d) \) is both \( c_2 \)-quasiconvex and \( c_2 \)-annular convex, then \( (\Omega, d) \) is \( c \)-uniform with \( c = c(c_1, c_2) \);

4. If \( d'_0 > 0 \), \( p \in \partial \Omega \), and \( (\Omega, d) \) is \( c \)-uniform, then \( (\Omega, d_p) \) is \( c' \)-uniform with \( c' = c'(c) \).

Under the assumptions of Theorem 4.1 (2), \((\Omega, k_{\Omega, p})\) is \( \delta' \)-hyperbolic for some \( \delta' \) whenever \((\Omega, k_{\Omega})\) is \( \delta \)-hyperbolic. In general, one cannot control \( \delta' \) in terms of \( \delta \) and \( c \) alone.

**Proposition 4.2.** Let \( (X, d) \) be a \( c \)-quasiconvex and \( c \)-annular convex proper metric space, \( \Omega \subset X \) a bounded rectifiably connected open subset of \( X \) such that \( d_0 = \text{diam}(\Omega, d) \) is finite and \( d'_0 = \text{diam}(\partial \Omega, d) \) is positive. If \( (\Omega, k_{\Omega}) \) is \( \delta \)-hyperbolic and \( p \in \partial \Omega \), then \((\Omega, k_{\Omega, p})\) is \( \delta' \)-hyperbolic with \( \delta' \) depending only on \( c \) and \( \delta \).

To prove this proposition we need the following preliminary results.

Let \( L \geq 1 \) and \( A \geq 0 \). A map \( f : X \to Y \) between two metric spaces is an \((L, A)\)-quasiisometry if the following two conditions are satisfied:

1. \( d_X(x_1, x_2)/L - A \leq d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2) + A \) holds for all \( x_1, x_2 \in X \);
(2) For each \( y \in Y \), there is some \( x \in X \) with \( d_Y(f(x), y) \leq A \).

By definition an \( L \)-bilipschitz map is an \((L, 0)\) quasiisometry, and an \((L, A)\)-quasigeodesic is an \((L, A)\)-quasiisometry onto its image. Observe that we do not require a quasiisometry to be continuous.

It is well-known that if \( f : Y_1 \to Y_2 \) is an \((L, A)\)-quasiisometry between geodesic spaces and \( Y_1 \) is \( \delta \)-hyperbolic, then \( Y_2 \) is \( \delta' \)-hyperbolic with \( \delta' = \delta'(\delta, L, A) \). Proposition 4.2 does not follow from this fact since the bilipschitz constant of the identity \( f : (\Omega, k_\Omega) \to (\Omega, k_{\Omega,p}) \) depends on the ratio \( d_0/d'_0 \). In fact, from Lemma 4.4 below we see that \( \text{diam}(K, k_\Omega) \leq 7c \) while it can be seen that \( \text{diam}(K, k_\Omega) \to \infty \) as \( d_0/d'_0 \to \infty \).

Now suppose \( X \) is a \( c \)-quasiconvex and \( c \)-annular convex proper metric space, \( \Omega \subset X \) is a rectifiably connected open subset of \( X \) with and \( 0 < d'_0 < \frac{d_0}{20c^2} < \infty \), and \( p \in \partial \Omega \). Then the boundary of \( \Omega \) in \((X, d_p)\) is \( \partial_p \Omega := \partial \Omega \setminus \{p\} \), and the boundary of \( \Omega \) in the induced quasihyperbolic metric \( k_{\Omega,p} \) is denoted \( \partial^p \Omega \). For \( x \in \Omega \), let \( \delta_p(x) = d_p(x, \partial_p \Omega) \). However, \( B = B(p, 10c^2d'_0) \) will denote the ball with respect to the original metric \( d \). Let \( K = \Omega \setminus B \) and \( S = \{ x \in \Omega : d(x, p) = 10c^2d'_0 \} \). By the definition of \( d'_0 \) and the assumption that \( d_0 \) is finite, observe that \( K \) is a compact subset of \( \Omega \) in both metrics. Let \( D_1 = \text{diam}(S, k_\Omega) \), \( D_2 = \text{diam}(S, k_{\Omega,p}) \) and \( d_2 = \text{diam}(K, k_{\Omega,p}) \). Since \( S \subset K \), we always have \( D_2 \leq d_2 \).

Lemma 4.3 (Lemma 3.9 of [BHX]). There is a constant \( L \) depending only on \( c \) such that for all \( x, y \in \Omega \setminus K \) we have \( k_{\Omega,p}(x, y) \leq L k_\Omega(x, y) + D_2 \) and \( k_\Omega(x, y) \leq L k_{\Omega,p}(x, y) + D_1 \).

Recall that we assume \( X \) to be both \( c \)-quasiconvex and \( c \)-annular convex. The length of a path \( \gamma \subset I_p(X) \) with respect to the \( d_p \)-metric shall be denoted \( \ell_p(\gamma) \).

Lemma 4.4. The inequalities \( D_1 \leq 4c^2 \) and \( d_2 \leq 7c \) hold.

Proof. Let \( x, y \in S \). Since \( X \) is \( c \)-annular convex, there is a path \( \gamma \in X \) joining \( x \) and \( y \) such that \( |\gamma| \subset B(p, 10c^3d'_0) \setminus B(p, 10cd'_0) \) and \( \ell(\gamma) \leq cd(x, y) \). Note that \( \delta_\Omega(z) \geq 5cd'_0 \) for all \( z \in |\gamma| \) and \( \gamma \subset \Omega \). Now

\[
\begin{align*}
k(x, y) &\leq \int_\gamma \frac{1}{\delta_\Omega(z)} |dz| \leq \int_\gamma \frac{1}{5cd'_0} |dz| = \frac{1}{5cd'_0} \ell(\gamma) \leq \frac{1}{5cd'_0} cd(x, y) \leq \frac{1}{5d'_0} 20c^2d'_0 = 4c^2.
\end{align*}
\]

Now we prove the second inequality. We first prove that whenever \( r \geq 10c^2d'_0 \), for every \( x, y \in (B(p, 2r) \setminus B(p, r)) \cap \Omega \),

\[
k_{\Omega,p}(x, y) \leq \frac{32c^3d'_0}{r}.
\]

Assume \( r \geq 10c^2d'_0 \) and let \( x, y \in (B(p, 2r) \setminus B(p, r/c)) \cap \Omega \). Since \( X \) is \( c \)-annular convex, there is a path \( |\gamma| \subset B(p, 2cr) \setminus B(p, r/c) \) connecting \( x \) and \( y \) such that \( \ell_d(\gamma) \leq cd(x, y) \). Note \( |\gamma| \subset \Omega \). For any \( z_1, z_2 \in |\gamma| \) we have by Theorem 4.1,

\[
d_p(z_1, z_2) \leq \frac{d(z_1, z_2)}{d(z_1, p)} \frac{d(z_2, p)}{d(z_2, p)} \leq d(z_1, z_2) \frac{c^2d(z_1, z_2)}{(r/c)^2} = \frac{c^2d(z_1, z_2)}{r^2}.
\]

It follows that

\[
\ell_p(\gamma) \leq \frac{c^2 \ell_d(\gamma)}{r^2} \leq \frac{c^2 \cdot cd(x, y)}{r^2} \leq \frac{c^3 \cdot 4r}{r^2} = 4c^3/r.
\]
On the other hand, as \( r \geq 10c^2d'_0 + 4 \) and \(|\gamma| \subset B(p, 2cr) \setminus B(p, r/c)\), we have \( d(z, w) \geq d(z, p) / 2 \) for all \( z \in |\gamma| \) and \( w \in \partial_p \Omega \). Hence by Theorem 4.1 again, for any \( w \in \partial_p \Omega \) and \( z \in |\gamma| \),
\[
d_p(z, w) \geq \frac{d(z, w)}{4d(z, p)d(w, p)} \geq \frac{1}{8d(w, p)} \geq \frac{1}{8d'_0}.
\]

It follows that \( \delta_p(z) \geq \frac{1}{8d'_0} \) for all \( z \in |\gamma| \). Consequently
\[
k_{\Omega, p}(x, y) \leq \int_{\gamma} \frac{1}{\delta_p(z)} ds_p \leq 8d'_0 L_p(\gamma) \leq 8d'_0 \cdot 4e^3 / r = \frac{32c^3d'_0}{r},
\]
where \( ds_p \) is the \( d_p \)-arc length element along \( \gamma \).

Set \( r_0 = 10c^2d'_0 \) and let \( n \geq 1 \) be the integer such that \( 2^{n-1}r_0 < d_0 \leq 2^n r_0 \). Then
\[
K \subset \bigcup_{i=1}^{n+1} \Omega \cap (B(p, 2^{i-1}r_0)) \setminus B(p, 2^{-1}r_0)).
\]
The inequality (5) now implies
\[
d_2 \leq \sum_{i=0}^{n} \frac{32c^3d'_0}{2^n r_0} \leq 2\frac{32c^3d'_0}{r_0} < 7c.
\]

Let \( \alpha : I \to \Omega \) be a geodesic with respect to the metric \( k_{\Omega, p} \) with \( I \) a closed (not necessarily compact) interval such that the endpoints of \( \alpha \) do not lie in \( K \). We define a map \( \alpha' : I \to (\Omega, k_{\Omega}) \) as follows. Let \( f : (\Omega, k_{\Omega}) \to (\Omega, k_{\Omega, p}) \) be the identity map. If \( |\alpha| \cap K = \emptyset \), then we let \( \alpha' = f^{-1} \circ \alpha \). If \( |\alpha| \cap K \neq \emptyset \), then let \( t_1 = \inf \alpha^{-1}(K) \) and \( t_2 = \sup \alpha^{-1}(K) \); observe that \( \alpha(t_1), \alpha(t_2) \in S \). Since \( \text{diam}(S, k_{\Omega, p}) = D_2 \), we have \( t_2 - t_1 \leq D_2 \). Let \( \alpha'(t) = f^{-1}(\alpha(t)) \) if \( t < t_1 \) or \( t > t_2 \), and \( \alpha'(t) = f^{-1}(\alpha(t_1)) \) if \( t \in [t_1, t_2] \). Similarly, given any geodesic \( \beta : I \to (\Omega, k_{\Omega}) \) whose endpoints do not lie in \( K \), we can define a map \( \beta' : I \to (\Omega, k_{\Omega, p}) \).

**Lemma 4.5.** The maps \( \alpha' \), \( \beta' \) are \((L, A)\)-quasigeodesics with respect to both metrics \( k_{\Omega} \) and \( k_{\Omega, p} \), with \( L \) the constant in Lemma 4.3 and \( A \) depends only on \( c \).

**Proof.** We only prove the claim for \( \alpha' \), as the proof for \( \beta' \) is similar. We use Lemma 4.3. Let \( s, t \in I \). First assume \( s, t \in I \setminus [t_1, t_2] \). Then \( \alpha'(s) = \alpha(s), \alpha'(t) = \alpha(t) \) and hence
\[
\frac{|s - t|}{L} - \frac{D_2}{L} = k_{\Omega, p}(\alpha(s), \alpha(t)) - \frac{D_2}{L} \leq k_{\Omega}(\alpha'(s), \alpha'(t)) \leq L k_{\Omega}(\alpha(s), \alpha(t)) + D_1 = L|s - t| + D_1.
\]

Next assume \( s, t \in [t_1, t_2] \). Then \( |s - t| \leq t_2 - t_1 \leq D_2 \) and \( \alpha'(s) = \alpha'(t) \). We therefore see that the above chain of inequalities is again satisfied. Finally assume \( s \in [t_1, t_2] \) and \( t \notin [t_1, t_2] \). Then
\[
k_{\Omega}(\alpha'(s), \alpha'(t)) = k_{\Omega}(\alpha'(t_1), \alpha'(t)) \leq L|t_1 - t| + D_1 \leq L|t_1 - s| + |s - t| + D_1 \leq L|s - t| + LD_2 + D_1,
\]
and
\[
k_{\Omega}(\alpha'(s), \alpha'(t)) = k_{\Omega}(\alpha'(t_1), \alpha'(t)) \geq \frac{|t_1 - t|}{L} - \frac{D_2}{L} \geq \frac{|s - t|}{L} - \frac{|s - t|}{L} - \frac{D_2}{L} \geq \frac{|s - t|}{L} - \frac{D_2}{L} - \frac{D_2}{L}.
\]

Now the lemma follows from Lemma 4.4. \( \Box \)
Proof of Proposition 4.2. The result follows from Theorem 4.1 (2) if $d_0 \leq 20c^2d_0'$. Hence without loss of generality, $d_0 > 20c^2d_0'$. The goal is to prove that all geodesic triangles in $(\Omega, k_{\Omega,p})$ are $\delta'$-thin for some $\delta' = \delta'(\delta, c)$.

Let $x_1, x_2, x_3 \in \Omega$ and setting $x_i := x_1$, let $a_i$ ($i = 1, 2, 3$) be a geodesic in $(\Omega, k_{\Omega,p})$ joining $x_i$ and $x_{i+1}$. We want some $\delta' = \delta'(\delta, c)$ such that $|a_1| \subset N_{\delta'}(|a_2| \cup |a_3|)$. We consider several cases.

Case 1: $x_1, x_2, x_3 \notin K$. By Lemma 4.5 we obtain an $(L, A)$-quasigeodesic $a'_1$ in $(\Omega, k_{\Omega})$ connecting $x_i$ and $x_{i+1}$. Fixing a geodesic $\beta_i$ in $(\Omega, k_{\Omega})$ joining $x_i$ and $x_{i+1}$, note by Lemma 3.3 that $HD_{k_{\Omega}}(|\beta_i|, |a'_i|) \leq c_1$ with $c_1 = c_1(\delta, L, A) = c_1(\delta, c)$. Let $x \in |a_1|$, and fix $y \in |a_1| \setminus K$ such that $k_{\Omega,p}(x, y) \leq D_2$. Necessarily $y \in |a'_1|$. Since $HD_{k_{\Omega}}(|\beta_i|, |a'_i|) \leq c_1$, there is some $y_1 \in |\beta_i|$ with $k_{\Omega}(y, y_1) \leq c_1$. By the $\delta$-hyperbolicity of $(\Omega, k_{\Omega})$ there is some $y_2 \in |\beta_2| \cup |\beta_3|$ with $k_{\Omega}(y_1, y_2) \leq \delta$. We may assume $y_2 \in |\beta_2|$. The fact $HD_{k_{\Omega}}(|\beta_2|, |\alpha_2|) \leq c_1$ implies that there is some $y_3 \in |\alpha_2|$ with $k_{\Omega}(y_2, y_3) \leq c_1$. Triangle inequality implies $k_{\Omega}(y, y_3) \leq 2c_1 + \delta$.

By Lemma 4.3 we have

$$k_{\Omega,p}(x, y_3) \leq k_{\Omega,p}(x, y) + k_{\Omega,p}(y, y_3) \leq D_2 + L k_{\Omega}(y, y_3) + D_2 \leq 2D_2 + L(2c_1 + \delta).$$

Since $y_3 \in |\alpha_2| \subset |\alpha_2|$, we have shown $x \in N_{\delta_1}(|\alpha_2| \cup |\alpha_3|)$ with $\delta_1 = 2D_2 + L(2c_1 + \delta)$.

Case 2: $x_1, x_2 \in K$. Since diam$(K, k_{\Omega,p}) = d_2$, we have $|a_1| \subset N_{d_2}(\{x_2\}) \subset N_{d_2}(|\alpha_2| \cup |\alpha_3|)$.

Case 3: $x_3 \in K$ and exactly one of $x_1, x_2$ lies in $K$, say $x_1 \in K$ and $x_2 \notin K$. Let $x_4$ be the first point on $\alpha_1$ (oriented from $x_2$ to $x_1$) that lies in $K$ and $x_3'$ the first point on $\alpha_2$ (oriented from $x_2$ to $x_3$) that lies in $K$. Let $\gamma'$ be a geodesic in $(\Omega, k_{\Omega,p})$ connecting $x_4$ and $x_3'$. Now by Case 1,

$$|a_1| \subset N_{d_2}(|\alpha_2[x_2, x_3]| \cup |\gamma'|) \subset N_{2d_2 + \delta_1}(|\alpha_2[x_2, x_3]| \cup |\gamma'|) \subset N_{2d_2 + \delta_1}(|\alpha_2|).$$

Case 4: $x_3 \notin K$ and exactly one of $x_1, x_2$ lies in $K$, say $x_1 \in K$ and $x_2 \notin K$. Let $x_3'$ be the first point on $\alpha_3$ (oriented from $x_2$ to $x_3$) that lies in $K$. Let $\gamma'$ be a geodesic in $(\Omega, k_{\Omega,p})$ joining $x_2$ and $x_3'$. Again Case 1 implies that $|\gamma'| \subset N_{d_1}(|\alpha_3[x_3, x_3']| \cup |\alpha_2|)$ (strictly speaking we need $x_3' \notin K$ in order to apply Case 1 here, but by the choice of $x_3'$ we may employ a limiting argument together with Case 1 to get the desired inclusion), and an application of Case 3 yields $|a_1| \subset N_{2d_2 + \delta_1}(|\alpha_3[x_1, x_3']| \cup |\gamma'|)$. It follows that

$$|a_1| \subset N_{2d_2 + 2\delta_1}(|\alpha_2| \cup |\alpha_3|).$$

Case 5: $x_1, x_2 \notin K$ and $x_3 \in K$. Let $x_4$ be the first point on $\alpha_3$ (oriented from $x_1$ to $x_3$) that lies in $K$. Let $\gamma'$ be a geodesic in $(\Omega, k_{\Omega,p})$ connecting $x_2$ and $x_4$. Case 1 implies that $|a_1| \subset N_{\delta}(|\alpha_3[x_1, x_4]| \cup |\gamma'|)$, and Case 3 implies that $|\gamma'| \subset N_{2d_2 + \delta_1}(|\alpha_2|)$. It follows that $|a_1| \subset N_{2d_2 + 2\delta_1}(|\alpha_2| \cup |\alpha_3|)$.

We shall also need the following construction of Bonk–Kleiner [BK].

Let $(X, d)$ be an unbounded metric space and $p \in X$. Let $S_p(X) = X \cup \{\infty\}$, where $\infty$ is a point not in $X$. We define a function $s_p : S_p(X) \times S_p(X) \to [0, \infty)$ as follows:

$$s_p(x, y) = \begin{cases} \frac{d(x, y)}{1 + d(x, p) |1 + d(y, p)|} & \text{if } x, y \in X, \\ \frac{1}{1 + d(x, p)} & \text{if } x \in X \text{ and } y = \infty, \\ 0 & \text{if } x = \infty = y. \end{cases}$$
In analogy to the construction of the metric $d_p$ for the inversion, we construct the metric $\hat{d}_p$ on $S_p(X)$ by the formula

$$\hat{d}_p(x,y) := \inf \Sigma_{i=0}^{k-1} s_p(x_i, x_{i+1}),$$

where the infimum is taken over all finite sequences of points $x_0, \ldots, x_k \in S_p(X)$ with $x_0 = x$ and $x_k = y$. It was shown in [BK] that $\hat{d}_p$ is a metric on $S_p(X)$ with

$$\frac{1}{4} s_p(x,y) \leq \hat{d}_p(x,y) \leq s_p(x,y) \quad \text{for} \quad x,y \in S_p(X).$$

Furthermore, the identity map $f : (X,d) \to (X,\hat{d}_p)$ is an $\eta$-quasimöbius homeomorphism with $\eta(t) = 16t$.

**Theorem 4.6 ([BHX]).** Let $(X,d)$ be an unbounded proper metric space, $\Omega \subset X$ a rectifiably connected open subset of $X$, and $p \in \partial \Omega$. We denote by $\tilde{k}_{\Omega,p}$ the quasihyperbolic metric on $\Omega$ induced by the metric $\hat{d}_p$. Suppose $\text{diam}(\Omega,d) = \infty$.

1. If $\Omega$ is locally $c$-quasiconvex, then the identity map $f : (\Omega,k_{\Omega}) \to (\Omega,\tilde{k}_{\Omega,p})$ is $M$-bilipschitz with $M$ depending only on $c$;

2. If $(\Omega,d)$ is $c$-uniform, then $(\Omega,\hat{d}_p)$ is $c'$-uniform with $c'$ depending only on $c$;

3. If $(X,d)$ is $c$-quasiconvex and $c$-annular convex, then $(S_p(X),\hat{d}_p)$ is $c'$-quasiconvex and $c'$-annular convex with $c'$ depending only on $c$;

4. If $(\Omega,\hat{d}_p)$ is $c$-uniform, then $(\Omega,d)$ is $c'$-uniform with $c'$ depending only on $c$.

5 **Boundary maps of quasi-isometries**

Under the assumptions of Theorem 4.1 (2), $(\Omega,k_{\Omega,p})$ is Gromov hyperbolic and the extension of the identity map $(\Omega,k_{\Omega}) \to (\Omega,k_{\Omega,p})$ is $\eta$-quasimöbius for some $\eta$ whenever $(\Omega,k_{\Omega})$ is Gromov hyperbolic. In general, there is no control on $\eta$. In this section we provide a control on $\eta$ in the case $(X,d)$ is annular convex (Proposition 5.6).

It is well-known that if $f : Y_1 \to Y_2$ is an $(L,A)$-quasiisometry between geodesic spaces and $Y_1$ is $\delta$-hyperbolic, then the natural boundary map $(\partial^* Y_1, d_{y_1,\epsilon}) \to (\partial^* Y_2, d_{y_2,\epsilon})$ of $f$ is $\eta$-quasimöbius with $\eta$ depending only on $L, A$ and $\delta$, see Proposition 5.10. Proposition 5.6 does not follow from this general result since the bilipschitz constant of the identity map $f : (\Omega,k_{\Omega}) \to (\Omega,k_{\Omega,p})$ depends on the ratio $d_0/d'_0$, where $d_0 := \text{diam}(\Omega,d)$ and $d'_0 := \text{diam}(\partial \Omega,d)$. See also the remark after Proposition 4.2. We first study the cross ratio on the Gromov boundary of a Gromov hyperbolic space (Corollary 5.2).

Let $(Y,d)$ be a proper geodesic $\delta$-hyperbolic space and $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ a quadruple of distinct points in $\partial^* Y$. The signed distance $\text{sd}(Q)$ of $Q$ is the number

$$\text{sd}(Q) = \inf \{d([\xi_1,\xi_4], [\xi_2,\xi_3]) - d([\xi_1,\xi_3], [\xi_2,\xi_4])\},$$

where the infimum is taken over all geodesics $[\xi_i,\xi_j]$ joining $\xi_i$ and $\xi_j$. It follows from Theorem 3.4 and the definition of $\delta$-hyperbolicity that the Hausdorff distance between two
infinite geodesics with the same endpoints is at most 2δ. Hence for all geodesics \([\xi_i, \xi_j]\) joining \(\xi_i\) and \(\xi_j\),
\[
    sd(Q) \leq d([\xi_1, \xi_4], [\xi_2, \xi_3]) - d([\xi_1, \xi_3], [\xi_2, \xi_4]) \leq sd(Q) + 8\delta.
\]
For \(w \in Y\), the cross difference of \(Q\) based at \(w\) is:
\[
    cd_w(Q) = (\xi_1|\xi_4)_w + (\xi_2|\xi_3)_w - (\xi_1|\xi_3)_w - (\xi_2|\xi_4)_w.
\]
Note that for all quadruples \(Q\), each \(w \in Y\), and every \(0 < \epsilon \leq \epsilon_0(\delta)\),
\[
    e^{\epsilon_{cdw}(Q)} / 4 \leq cr(Q, d_{w, \epsilon}) \leq 4e^{\epsilon_{cdw}(Q)}.
\]
Moreover, if \(Y\) is a tree, then \(sd(Q) = cd_w(Q)\) for all \(w \in Y\) and all \(Q\). The following result shows that in a general \(\delta\)-hyperbolic geodesic space, \(sd(Q)\) and \(cd_w(Q)\) differ by at most a fixed multiple of \(\delta\). From Theorem 3.4 and the definition of \(\delta\)-hyperbolicity it follows that geodesic triangles in \(Y \cup \partial^r Y\) are 24\(\delta\)-thin.

**Lemma 5.1.** Let \((Y, d)\) be a \(\delta\)-hyperbolic space, \(w \in Y\), and \(Q = (\xi_1, \xi_2, \xi_3, \xi_4)\) a quadruple of distinct points in \(\partial^r Y\). Then \(|cd_w(Q) - sd(Q)| \leq 430\delta\).

**Proof.** Fix \(w \in Y\) and we choose geodesic rays \([w, \xi_i], 1 \leq i \leq 4\), geodesics \([\xi_i, \xi_j]\), and let \(X = \bigcup\{[w, \xi_i]\}\). By Theorem 3.4 there is a tree \(T(X)\) and a map \(u : X \rightarrow T(X)\) with the properties stated in Theorem 3.4. Let \(w' = u(w)\) and \(\xi'_i \in \partial^r T(X)\) be such that \(u([w, \xi_i])\) is an isometry onto \([w', \xi'_i]\). Let \(x'_{ij} \in T(X)\) be the unique point with \([w', x'_{ij}] = [w', \xi'_i] \cap [w', \xi'_j]\) \((x'_{ij} = x'_{ji})\), and let \(x_{ij} \in w\xi_i\) be such that \(u(x_{ij}) = x'_{ij}\) \((x_{ij}\) may not equal \(x'_{ji}\)).

We can find sequences \(x_k \in [w, \xi_i]\) converging to \(\xi_i\) and \(y_k \in [w, \xi_j]\) converging to \(\xi_j\). Since \(T(X)\) is a tree and hence \((u(x_k)|u(y_k))_{w'} = d(w', x'_{ij})\), the properties of \(u\) from Theorem 3.4 imply that
\[
    d(w', x'_{ij}) - 6\delta \leq (x_k|y_k)_w \leq d(w', x'_{ij}).
\]
Using \((\xi_i|\xi_j)_w - 2\delta \leq \liminf_{k, l \rightarrow \infty}(x_k|y_l)_w \leq (\xi_i|\xi_j)_w\), we obtain
\[
    d(w', x'_{ij}) - 3\delta \leq (\xi_i|\xi_j)_w \leq d(w', x'_{ij}) + 2\delta.
\]
It follows that with \(Q' = (\xi'_1, \xi'_2, \xi'_3, \xi'_4)\),
\[
    cd_{w'}(Q') - 10\delta \leq cd_w(Q) \leq cd_{w'}(Q') + 10\delta, \quad (6)
\]
We next show that \(HD([\xi_i, \xi_j], [x_{ij}, \xi_i] \cup [x_{ji}, \xi_j]) \leq 100\delta\). To this end, let \(y_{ij} \in [x_{ij}, \xi_i]\) be such that \(d(x_{ij}, y_{ij}) = 25\delta\). The properties of \(u\) imply that \(d(y_{ij}, [w, \xi_j]) \geq 25\delta\). Since the triangle \([w, \xi_i] \cup [w, \xi_j] \cup [\xi_i, \xi_j]\) is \(24\delta\)-thin, there is some point \(z_{ij} \in [\xi_i, \xi_j]\) such that \(d(y_{ij}, z_{ij}) \leq 24\delta\). Consequently, the fact that \([y_{ij}, z_{ij}] \cup [y_{ij}, \xi_i] \cup [z_{ij}, \xi_j]\) is \(24\delta\)-thin implies that \(HD([y_{ij}, \xi_i], [z_{ij}, \xi_i]) \leq 48\delta\). Similarly \(HD([y_{ji}, \xi_i], [z_{ji}, \xi_i]) \leq 48\delta\). Since \(d(x_{ij}, x_{ji}) \leq 6\delta\), the triangle inequality implies that \(d(z_{ij}, z_{ji}) \leq 104\delta\). It follows that
\[
    HD([\xi_i, \xi_j], [x_{ij}, \xi_i] \cup [x_{ji}, \xi_j]) \leq 48\delta + 52\delta = 100\delta, \quad (7)
\]
For \(\{i, j, k, l\} = \{1, 2, 3, 4\}\), we choose \(p_{ij} \in [\xi_i, \xi_j]\) and \(p_{kl} \in [\xi_k, \xi_l]\) with
\[
    d(p_{ij}, p_{kl}) = d([\xi_i, \xi_j], [\xi_k, \xi_l]).
\]

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Inequality (7) implies that there are \( q_{ij} \in [x_{ij}, \xi_i] \cup [x_{ji}, \xi_j] \) and \( q_{kl} \in [x_{kl}, \xi_k] \cup [x_{lk}, \xi_l] \) such that \( |d(p_{ij}, p_{kl}) - d(q_{ij}, q_{kl})| \leq 200\delta \). By the properties of \( u \),
\[
    d(q_{ij}, q_{kl}) - 6\delta \leq d(u(q_{ij}), u(q_{kl})) \leq d(q_{ij}, q_{kl}).
\]
Since \( u(q_{ij}) \in [\xi'_i, \xi'_j] \) and \( u(q_{kl}) \in [\xi'_k, \xi'_l] \), we have \( d(u(q_{ij}), u(q_{kl})) \geq d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l]) \). Combining the above inequalities we obtain
\[
    d([\xi_i, \xi_j], [\xi_k, \xi_l]) \geq d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l]) - 200\delta. \tag{8}
\]
On the other hand, there exist points \( r'_{ij} \in [\xi'_i, \xi'_j] \) and \( r'_{kl} \in [\xi'_k, \xi'_l] \) such that
\[
    d(r'_{ij}, r'_{kl}) = d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l]).
\]
Observe that \( u([x_{ij}, \xi_i] \cup [x_{ji}, \xi_j]) = [\xi'_i, \xi'_j] \). Hence there exists \( r_{ij} \in [x_{ij}, \xi_i] \cup [x_{ji}, \xi_j] \) with \( u(r_{ij}) = r'_{ij} \). Similarly there is a point \( r_{kl} \in [x_{kl}, \xi_k] \cup [x_{lk}, \xi_l] \) with \( u(r_{kl}) = r'_{kl} \). The properties of \( u \) gives \( d(r_{ij}, r_{kl}) - 6\delta \leq d(r'_{ij}, r'_{kl}) \leq d(r_{ij}, r_{kl}) \). By inequality (7), there is a point \( w_{ij} \in [\xi_i, \xi_j] \) with \( d(w_{ij}, r_{ij}) \leq 100\delta \). Similarly there is some \( w_{kl} \in [\xi_k, \xi_l] \) with \( d(w_{kl}, r_{kl}) \leq 100\delta \). Thus,
\[
    d([\xi_i, \xi_j], [\xi_k, \xi_l]) \leq d(w_{ij}, w_{kl}) \leq d(r_{ij}, r_{kl}) + 200\delta
    \leq d(r'_{ij}, r'_{kl}) + 206\delta = d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l]) + 206\delta,
\]
whence by inequality (8),
\[
    |d([\xi_i, \xi_j], [\xi_k, \xi_l]) - d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l])| \leq 206\delta. \tag{9}
\]
Also recall that
\[
    sd(Q) - 8\delta \leq d([\xi_1, \xi_4], [\xi_2, \xi_3]) - d([\xi_1, \xi_3], [\xi_2, \xi_4]) \leq sd(Q) + 8\delta.
\]
It follows that \( |sd(Q) - sd(Q')| \leq 420\delta \). Since \( T(X) \) is a tree, we have \( cd_{w'}(Q') = sd(Q') \). Therefore by inequality (6), \( |cd_{w}(Q) - sd(Q)| \leq 430\delta \). \( \square \)

Since \( T(X) \) is a tree, at least one of \( d([\xi'_1, \xi'_4], [\xi'_2, \xi'_3]) \), \( d([\xi'_1, \xi'_3], [\xi'_2, \xi'_4]) \) is 0. Hence it follows from inequality (9) that
\[
    \min \left\{ d([\xi_1, \xi_4], [\xi_2, \xi_3]), d([\xi_1, \xi_3], [\xi_2, \xi_4]) \right\} \leq 206\delta. \tag{10}
\]

**Corollary 5.2.** Let \((Y, d)\) be a proper geodesic \( \delta \)-hyperbolic space, and set \( c_0 = 4e^{86} \). Then for each \( w \in Y \), all \( 0 < \epsilon \leq c_0(\delta) \), and all quadruple \( Q \) of distinct points in \( \partial^* Y \),
\[
    \frac{e^{\epsilon \cdot sd(Q)}}{c_0} \leq cr(Q, d_{w, \epsilon}) \leq c_0 e^{\epsilon \cdot sd(Q)}.
\]

**Proof.** Recall \( \epsilon(\delta) = \min\{1, \frac{1}{5\delta}\} \). The corollary now follows from Lemma 5.1 and the inequality
\[
    \frac{1}{4} e^{\epsilon \cdot cd_{w}(Q)} \leq cr(Q, d_{w, \epsilon}) \leq 4 e^{\epsilon \cdot cd_{w}(Q)}.
\]
Corollary 5.3. Let \( c > 0 \), \((Y, d)\) a proper geodesic \( \delta \)-hyperbolic space, \( x \in Y \), \( 0 < \epsilon \leq \epsilon_0(\delta) \), and \( Q = (\xi_1, \xi_2, \xi_3, \xi_4) \) a quadruple of distinct points in \( \partial^* Y \).

(1) If \( cr(Q, d_{x,\epsilon}) \leq c \), then for all geodesics \([\xi_i, \xi_j]\) joining \( \xi_i \) and \( \xi_j \) for \( 1 \leq i, j \leq 4 \),

\[
d([\xi_1, \xi_4], [\xi_2, \xi_3]) \leq c' = c'(c, \epsilon, \delta).
\]

(2) If \( d([\xi_1, \xi_4], [\xi_2, \xi_3]) \leq c \) for some geodesics \([\xi_i, \xi_j]\) joining \( \xi_i \) and \( \xi_j \), \( 1 \leq i, j \leq 4 \), then

\[
cr(Q, d_{x,\epsilon}) \leq c' = c'(c).
\]

Proof. We first prove (1). By hypothesis \( cr(Q, d_{x,\epsilon}) \leq c \). Therefore, by Corollary 5.2 we have \( sd(Q) \leq \log(c_0 c)/\epsilon \). Let \([\xi_i, \xi_j]\) be a geodesic with endpoints \( \xi_i \) and \( \xi_j \) for \( 1 \leq i, j \leq 4 \). With the aid of inequality (10) the claim follows from the fact (see the discussion following the definition of \( sd(Q) \)) that

\[
d([\xi_1, \xi_4], [\xi_2, \xi_3]) - d([\xi_1, \xi_3], [\xi_2, \xi_4]) \leq sd(Q) + 8\delta.
\]

Now we prove (2). Suppose that \( d([\xi_1, \xi_4], [\xi_2, \xi_3]) \leq c \). Then \( sd(Q) \leq c \). Since \( \epsilon \leq 1 \), by Corollary 5.2 we have \( cr(Q, d_{x,\epsilon}) \leq c_0 e^{c \epsilon} \leq c_0 e^{c} \).

Corollary 5.4. Let \((Y, d)\) be \( \delta \)-hyperbolic and \( w_1, w_2 \in Y \). Then for any \( 0 < \epsilon \leq \epsilon_0(\delta) \), the identity map \( (\partial^* Y, d_{w_1,\epsilon}) \to (\partial^* Y, d_{w_2,\epsilon}) \) is \( \eta \)-quasimöbius with \( \eta(t) = 16e^{172} t = c_0^2 t \).

Proof. By Corollary 5.2,

\[
 cr(Q, d_{w_2,\epsilon}) \leq c_0 e^{\epsilon sd(Q)} \leq c_0^2 cr(Q, d_{w_1,\epsilon}).
\]

Corollary 5.5. Let \((Y, d)\) be \( \delta \)-hyperbolic and \( 0 < \epsilon_1, \epsilon_2 \leq \epsilon_0(\delta) \). Then for any \( w \in Y \), the identity map \( (\partial^* Y, d_{w,\epsilon_1}) \to (\partial^* Y, d_{w,\epsilon_2}) \) is \( \eta \)-quasimöbius with \( \eta(t) = 4(1 + \frac{\epsilon_2}{\epsilon_1}) t^{\frac{\epsilon_2}{\epsilon_1}} \).

Proof. let \( t = cr(Q, d_{w,\epsilon_1}) \) for any quadruple \( Q \) of distinct points from \( \partial^* Y \). Then

\[
 cr(Q, d_{w,\epsilon_2}) \leq 4e^{cr_{d,w}(Q)} = 4(e^{c_{d,w}(Q)} \eta_1) \leq 4(4t)^{\frac{\epsilon_2}{\epsilon_1}} = \eta(t).
\]

We now bring the construction of inversion back into the picture. By Proposition 4.2 \((\Omega, k_{\Omega,p})\) is \( \delta' \)-hyperbolic with \( \delta' \) depending only on \( c \) and \( \delta \). Theorem 4.1 implies that the identity map \( f : (\Omega, k_\Omega) \to (\Omega, k_{\Omega,p}) \) is bilipschitz, hence it has a natural extension \( f' : (\partial^* \Omega, d_{w,\epsilon}) \to (\partial^* \Omega, d_{w,\epsilon}') \) to the associated boundaries for all \( w, w' \in \Omega \), and this extension is a homeomorphism.

Proposition 5.6. Suppose \((X, d)\) is a \( c \)-quasiconvex and \( c \)-annular convex proper metric space, \( \Omega \subset X \) a domain in \( X \), and \( p \in \partial \Omega \). Suppose also that \( d_0 := \text{diam}(\Omega, d) < \infty \) and \( d'_0 := \text{diam}(\partial \Omega, d) > 0 \). If \((\Omega, k_{\Omega})\) is \( \delta \)-hyperbolic, then \( f : (\partial^* \Omega, d_{w,\epsilon}) \to (\partial^* \Omega, d_{w,\epsilon}') \) is \( \eta \)-quasimöbius with \( \eta \) depending only on \( \delta \) and \( c \).
By an abuse of notation, for $\xi \in \partial_p^* \Omega$ we denote $\partial f(\xi)$ also by $\xi$. Let $K$ be the compact set given in Section 4 if $d_0 > 20c^2d_0'$, and $K = \emptyset$ if $d_0 \leq 20c^2d_0'$. The proof of Proposition 5.6 is achieved by combining Lemmas 5.7 through 5.9.

Lemma 5.7. There exists a constant $A' = A'(\delta, c)$ such that

$$
\frac{1}{L} k_\Omega([\xi_i, \xi_j], [\xi_k, \xi_l]) - A' \leq k_{\Omega,p}([\xi_i, \xi_j], [\xi_k, \xi_l]) \leq L k_\Omega([\xi_i, \xi_j], [\xi_k, \xi_l]) + A'
$$

for every quadruple $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ of distinct points in $\partial^* \Omega$, geodesics $[\xi_i, \xi_j]$ in $(\Omega, \kappa_\Omega)$ joining $\xi_i$ and $\xi_j$, and geodesics $[\xi_k, \xi_l]$ in $(\Omega, \kappa_{\Omega,p})$ joining $\xi_k$ and $\xi_l$. Here $L$ is the constant given by Lemma 4.3.

Proof. Let $q_{ij} \in [\xi_i, \xi_j]$ and $q_{kl} \in [\xi_k, \xi_l]$ such that $k_\Omega(q_{ij}, q_{kl}) = k_\Omega([\xi_i, \xi_j], [\xi_k, \xi_l])$. Since $\text{diam}(S, \kappa_\Omega) = D_1$ and if the geodesics $[\xi_i, \xi_j], [\xi_k, \xi_l]$ intersect $K$ then they both pass through $S$, there exist points $w_{ij} \in [\xi_i, \xi_j] \setminus K$ and $w_{kl} \in [\xi_k, \xi_l] \setminus K$ such that $k_\Omega(q_{ij}, w_{ij}) \leq D_1$ and $k_\Omega(q_{kl}, w_{kl}) \leq D_1$ (if either of the two geodesics, say $[\xi_i, \xi_j]$, does not intersect $K$, then we can choose $w_{ij} = q_{ij}$). Given the geodesics $\beta_{ij} = [\xi_i, \xi_j]$, consider the maps $\beta'_{ij}$ given in the paragraph before Lemma 4.5. By Lemma 4.5 both $\beta'_{ij}$ and $\beta'_{kl}$ are seen to be $(L, A)$-quasigeodesics in $(\Omega, \kappa_{\Omega,p})$, with $L$ and $A$ depending solely on $c$. By Theorem 4.1, the Gromov boundary points that are endpoints of $\beta'_{ij}$ given by Lemma 3.3 are the same as $\xi_i = \partial f(\xi_i)$ and $\xi_j = \partial f(\xi_j)$. From Lemma 3.3 and the fact that geodesic triangles in $(\Omega \cup \partial_p^* \Omega, \kappa_{\Omega,p})$ are $24\delta'$ thin it follows that $HD_{k_{\Omega,p}}(\beta'_{ij}, [\xi_i, \xi_j]) \leq b_1$ and $HD_{k_{\Omega,p}}(\beta'_{kl}, [\xi_k, \xi_l]) \leq b_1$, where $b_1 = b_1(\delta', L, A) = b_1(\delta, c)$. Hence we find two points $z_{ij} \in [\xi_i, \xi_j]$ and $z_{kl} \in [\xi_k, \xi_l]$ with $k_{\Omega,p}(w_{ij}, z_{ij}), k_{\Omega,p}(w_{kl}, z_{kl}) \leq b_1$. Now we have

$$
k_{\Omega,p}([\xi_i, \xi_j], [\xi_k, \xi_l]) \leq k_{\Omega,p}(z_{ij}, z_{kl}) \leq k_{\Omega,p}(w_{ij}, w_{kl}) + 2b_1 \leq L k_\Omega(w_{ij}, w_{kl}) + D_2 + 2b_1 \leq L \{k_\Omega(q_{ij}, q_{kl}) + 2D_1\} + D_2 + 2b_1 = L k_\Omega([\xi_i, \xi_j], [\xi_k, \xi_l]) + (D_2 + 2b_1 + 2LD_1).
$$

The second inequality can be proven in a similar manner. □

Let $Q' = (\partial f(\xi_1), \partial f(\xi_2), \partial f(\xi_3), \partial f(\xi_4))$ for each quadruple $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ of distinct points in $\partial^* \Omega$.

Lemma 5.8. There exists a constant $b_2 = b_2(\delta, c)$ with the property that for every quadruple $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ of distinct points in $\partial^* \Omega$,

1. if $sd(Q) \geq 0$, then $sd(Q') \leq L \cdot sd(Q) + b_2$;
2. if $sd(Q) \leq 0$, then $sd(Q') \leq sd(Q)/L + b_2$.

Proof. We first prove (1). Assume that $sd(Q) \geq 0$. Recall

$$
sd(Q) - 8\delta \leq k_\Omega([\xi_1, \xi_4], [\xi_2, \xi_3]) - k_\Omega([\xi_1, \xi_3], [\xi_2, \xi_4]) \leq sd(Q) + 8\delta.
$$

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Since by inequality (10)
\[
\min \left\{ k_\Omega([\xi_1, \xi_4], [\xi_2, \xi_3]), \; k_\Omega([\xi_1, \xi_3], [\xi_2, \xi_4]) \right\} \leq 206\delta,
\]
we have \( k_\Omega([\xi_1, \xi_3], [\xi_2, \xi_4]) \leq 214\delta \). Now Lemma 5.7 implies that
\[
sd(Q') \leq k_{\Omega, p}([\tilde{\xi}_1, \tilde{\xi}_4], [\tilde{\xi}_2, \tilde{\xi}_3]) - k_{\Omega, p}([\tilde{\xi}_1, \tilde{\xi}_3], [\tilde{\xi}_2, \tilde{\xi}_4]) + 8\delta' \\
\leq L k_\Omega([\xi_1, \xi_4], [\xi_2, \xi_3]) + A' + 8\delta' \\
= L (sd(Q) + k_\Omega([\xi_1, \xi_3], [\xi_2, \xi_4]) + 8\delta) + A' + 8\delta' \\
\leq L \cdot sd(Q) + 222 L \delta + A' + 8\delta'.
\]
The proof of (2) is similar to that of (1). If \( sd(Q) \leq 0 \), then \( k_\Omega([\xi_1, \xi_4], [\xi_2, \xi_3]) \leq 214\delta \), and therefore Lemma 5.7 implies
\[
sd(Q') \leq k_{\Omega, p}([\tilde{\xi}_1, \tilde{\xi}_4], [\tilde{\xi}_2, \tilde{\xi}_3]) - k_{\Omega, p}([\tilde{\xi}_1, \tilde{\xi}_3], [\tilde{\xi}_2, \tilde{\xi}_4]) + 8\delta' \\
\leq L k_\Omega([\xi_1, \xi_4], [\xi_2, \xi_3]) + A' - \frac{1}{L} k_\Omega([\xi_1, \xi_3], [\xi_2, \xi_4]) + A' + 8\delta' \\
\leq 214 L \delta + 2A' + 8\delta' + \frac{1}{L} \{ k_\Omega([\xi_1, \xi_4], [\xi_2, \xi_3]) - k_\Omega([\xi_1, \xi_3], [\xi_2, \xi_4]) \} \\
\leq 214 L \delta + 2A' + 8\delta' + \frac{1}{L} (sd(Q) + 8\delta) \\
= \frac{1}{L} sd(Q) + 214L \delta + 2A' + 8\delta' + 8\delta/L.
\]

Let \( w, w' \in \Omega \) and \( 0 < \epsilon \leq \epsilon_0(\delta), \epsilon_0(\delta') \). For a quadruple \( Q = (\xi_1, \xi_2, \xi_3, \xi_4) \) of distinct points in \( \partial^* \Omega \), set \( cr(Q) = cr(Q, d_{w, \epsilon}) \) and \( cr(Q') = cr(Q', d_{w', \epsilon}) \).

**Lemma 5.9.** The map \( \partial f : (\partial^* \Omega, d_{w, \epsilon}) \to (\partial^* \Omega, d_{w', \epsilon}) \) is \( \eta \)-quasimöbius for some \( \eta = \eta(\delta, c) \).

**Proof.** Let \( Q = (\xi_1, \xi_2, \xi_3, \xi_4) \). Recall that \( \epsilon \leq \epsilon_0(\delta) \leq 1 \). If \( sd(Q) \geq 0 \), then by Lemma 5.8, \( sd(Q') \leq L sd(Q) + b_2 \). Let \( c_0 = 4\epsilon^{86} \). It follows from Corollary 5.2 that
\[
\frac{cr(Q)}{cr(Q')} \leq c_0 e^{\frac{\epsilon}{L} (sd(Q) + b_2)} = c_0 e^{\frac{\epsilon}{L} (sd(Q))^L} \\
\leq c_0 e^{\frac{\epsilon}{L} (cr(Q))^L} \\
= c_0^L e^{\frac{\epsilon}{L} (cr(Q))^L} \leq c_0^{L+1} e^{\frac{\epsilon}{L} (cr(Q))^L}.
\]

If \( sd(Q) \leq 0 \), then \( sd(Q') \leq sd(Q)/L + b_2 \). It follows that
\[
\frac{cr(Q)}{cr(Q')} \leq c_0 e^{\frac{\epsilon}{L} (sd(Q) + b_2)} = c_0 e^{\frac{\epsilon}{L} (sd(Q))^L} \leq c_0 e^{\frac{\epsilon}{L} (cr(Q))^L} \\
= c_0^{L+1} e^{\frac{\epsilon}{L} (cr(Q))^L} \leq c_0^{L+1} e^{\frac{\epsilon}{L} (cr(Q))^L}.
\]

Note that \( cr(Q) \to 0 \) as \( sd(Q) \to -\infty \).
The proof of Proposition 5.6 can easily be generalized to show the following:

**Proposition 5.10.** Let $f : X \to Y$ be an $(L,A)$-quasiisometry between two proper geodesic metric spaces. If $X$ is $\delta$-hyperbolic, then $Y$ is $\delta'$-hyperbolic with $\delta' = \delta'(\delta, L, A)$ and the natural map $\partial f : (\partial^* X, d_{x,\varepsilon}) \to (\partial^* Y, d_{y,\varepsilon})$ with $x \in X$, $y \in Y$, is $\eta'$-quasimöbius with $\eta = \eta(L, A, \delta)$.

### 6 Necessity

In this section we prove that a uniform domain is Gromov hyperbolic with respect to the quasihyperbolic metric and that the natural map exists and is quasimöbius; this result is quantitative. We first explain the notion of natural map.

Let $(X,d)$ be a proper metric space, $X'$ the one point compactification $X \cup \{\infty\}$ of $X$ if $X$ is unbounded, and let $X' = X$ if $X$ is bounded. Let $Q = (x_1, x_2, x_3, x_4)$ be a quadruple of distinct points in $X'$. The cross ratio $cr(Q, d)$ is defined as in Section 4 if all $x_i \in X$, and if one of the $x_i$ is $\infty$, then $cr(Q, d)$ is obtained from the usual definition by canceling the terms involving $\infty$. For example, if $x_1 = \infty$, then

$$cr(Q, d) = \frac{d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)} = \frac{d(x_2, x_4)}{d(x_2, x_3)}.$$  

If $\Omega \subset X$ is a rectifiably connected open subset with $\partial \Omega \neq \emptyset$, let $\partial' \Omega$ be the boundary of $\Omega$ in $X'$. Suppose $(\Omega, k_{\Omega})$ is Gromov hyperbolic. If the identity map $(\Omega, k_{\Omega}) \to (\Omega, d_{\Omega})$ has a continuous extension from the Gromov closure $\overline{\Omega} = \Omega \cup \partial^* \Omega$ of $(\Omega, k)$ into $X'$, then the restriction of this extension to the Gromov boundary, $\partial' \Omega \to \partial \Omega$, is called a natural map of $\Omega$. Since $\Omega$ is dense in the Gromov closure, the natural map is unique if it exists.

We note that if $(X,d)$ is unbounded, then for any $p \in X$ the metric space $(S_p(X), d_p)$ is homeomorphic to the one point compactification $X'$. So by a natural map $\phi : \partial^* \Omega \to \partial' \Omega$ we mean the continuous extension to the Gromov boundary of the identity map $(\Omega, k_{\Omega}) \to (\Omega, d_{\Omega})$. Since the identity map $(X,d) \to (X, d_p)$ is $\eta_0$-quasimöbius with $\eta_0(t) = 16t$, a natural map $(\partial^* \Omega, d_{x,\varepsilon}) \to (\partial' \Omega, d)$ exists and is $\eta$-quasimöbius if and only if the natural map $(\partial^* \Omega, d_{x,\varepsilon}) \to (\partial' \Omega, d_p)$ exists and is $\eta'$-quasimöbius, with $\eta$ and $\eta'$ depending only on each other.

Suppose there is a metric space $(Y, \rho)$ and $\Omega \subset Y$ a bounded rectifiably connected open subset with $\partial \Omega$ (the metric boundary of $\Omega$ in $(Y, \rho)$) containing at least two points, and $p \in \partial \Omega$ such that $X = I_p(Y)$ with $d = d_p$; that is, the metric space $(X,d)$ is the “inversion” of $(Y, \rho)$ at $p$. Then $\Omega$ is unbounded in $(X,d)$. We note that $X'$ and $Y'$ are homeomorphic. If $(\Omega, k_{\Omega})$ is Gromov hyperbolic and $\phi : \partial^* \Omega \to \partial' \Omega$ is a natural map, then by composing $\phi$ with the identification of $X'$ and $Y'$, we obtain another natural map $\phi' : \partial^* \Omega \to \partial^* \Omega$. Since the identity map $(Y \setminus \{p\}, \rho) \to (X,d)$ is $\eta_0$-quasimöbius, we see that a natural map $\phi : (\partial^* \Omega, d_{x,\varepsilon}) \to (\partial' \Omega, d)$ is $\eta$-quasimöbius if and only if the natural map $(\partial^* \Omega, d_{x,\varepsilon}) \to (\partial^* \Omega, \rho)$ is $\eta'$-quasimöbius, with $\eta$ and $\eta'$ depending only on each other.

**Theorem 6.1 (Theorem 3.6 of [BHK]).** Let $(X,d)$ be a proper metric space and $\Omega \subset X$ a $c$-uniform domain. Then $(\Omega, k_{\Omega})$ is a geodesic $\delta$-hyperbolic space with $\delta = \delta(c)$. If $\Omega$ is
bounded, then for each \( w \in \Omega \) and all \( 0 < \epsilon \leq \epsilon_0(\delta) \) the natural map \( \phi : (\partial^* \Omega, d_{w,\epsilon}) \to (\partial \Omega, d) \) exists and is \( \eta \)-quasimöbius with \( \eta = \eta(c, \epsilon) \).

If we choose \( \epsilon = \epsilon_0(\delta) = \epsilon_0(c) \) for the visual metric \( d_{w,\epsilon} \), then the homeomorphism \( \eta \) in Theorem 6.1 depends only on \( c \).

**Theorem 6.2.** Let \( X \) be a proper metric space and \( \Omega \subset X \) a \( c \)-uniform domain. There exists a constant \( \epsilon_1(c) > 0 \) such that for every \( w \in \Omega \) and \( 0 < \epsilon \leq \epsilon_1(c) \), the natural map 
\[
\phi : (\partial^* \Omega, d_{w,\epsilon}) \to (\partial' \Omega, d) 
\]
exists and is \( \eta \)-quasimöbius with \( \eta = \eta(c, \epsilon) \).

**Proof.** By Theorem 6.1, it only remains to consider the case when \( \Omega \) is unbounded. Suppose that \( \Omega \) is an unbounded \( c \)-uniform domain. Fix \( p \in \partial \Omega \) and consider the compact metric space \( (S_p(X), \hat{d}_p) \). By Theorem 4.6 (2), \( (\Omega, \hat{d}_p) \) is \( c_1 \)-uniform with \( c_1 = c_1(c) \). Let \( k_{\Omega, p} \) be the quasihyperbolic metric on \( \Omega \subset (S_p(X), \hat{d}_p) \). By Theorem 6.1, \( (\Omega, k_{\Omega, p}) \) is \( \delta_1 \)-hyperbolic with \( \delta_1 = \delta_1(c_1) = \delta_1(c) \), and therefore for any \( w \in \Omega \) and \( 0 < \epsilon \leq \epsilon_0(\delta_1) \), the natural map 
\[
\phi_1 : (\partial^* \Omega, d_{w,\epsilon}) \to (\partial \Omega, \hat{d}_p) 
\]
exists and is \( \eta_1 \)-quasimöbius with \( \eta_1 = \eta_1(c_1, \epsilon) = \eta_1(c, \epsilon) \).

On the other hand, Theorem 4.6 (1) implies that the identity map \( f : (\Omega, k_{\Omega}) \to (\Omega, k_{\Omega, p}) \) is \( M \)-bilipschitz with \( M = M(c) \). By Proposition 5.10, for any \( w \in \Omega \) and any \( \epsilon \) satisfying \( 0 < \epsilon \leq \epsilon_1(c) := \min\{\epsilon_0(\delta), \epsilon_0(\delta_1)\} \), the boundary map \( \partial f : (\partial^* \Omega, d_{w,\epsilon}) \to (\partial^* \Omega, d'_{w,\epsilon}) \) is \( \eta_2 \)-quasimöbius with \( \eta_2 = \eta_2(\delta_1, M) = \eta_2(c) \). Hence there is an \( \eta \)-quasimöbius natural map 
\[
\phi = \phi_1 \circ \partial f : (\partial^* \Omega, d_{w,\epsilon}) \to (\partial' \Omega, \hat{d}_p) 
\]
with \( \eta = \eta_1 \circ \eta_2 \).

Again if we choose \( \epsilon = \epsilon_1(c) \), then the homeomorphism \( \eta \) in Theorem 6.2 depends only on \( c \).

### 7 Annulus points, arc points and starlikeness

In this section we recall the notion of annulus points and arc points, and show that each arc point lies on an anchor (Lemma 7.3) and that domains with large boundaries are starlike (Theorem 7.4).

The following definitions are from Chapter 7 of [BHK]. Let \( (X, d) \) be a proper geodesic space and \( \Omega \subset X \) a rectifiably connected open subset with \( \partial \Omega \neq \emptyset \).

**Definition 7.1.** Let \( 0 < \lambda \leq 1/2 \). A point \( x \in \Omega \) is said to be a \( \lambda \)-**annulus point** if there is a point \( a \in \partial \Omega \) with \( \delta_\Omega(x) = d(x, a) \) such that \( B(a, \delta_\Omega(x)/\lambda) \setminus B(a, \lambda \delta_\Omega(x)) \subset \Omega \). If \( x \in \Omega \) is not a \( \lambda \)-annulus point, then it is said to be a \( \lambda \)-**arc point**.

**Definition 7.2.** Let \( x_0 \in \Omega \) and \( c \geq 1 \). A path \( \gamma : a \leadsto b \) (with \( a, b \in \partial \Omega \) in \( \overline{\Omega} \)) is a \( c \)-**anchor** of \( x_0 \) if

1. \( x_0 \in |\gamma| \),
2. \( \ell_d(\gamma) \leq c \, d(a, b) \),
3. for every \( x \in |\gamma_{ax_0}| \) we have \( \ell_d(\gamma_{ax}) \leq c \, \delta_\Omega(x) \),
(iv) for every $x \in |\gamma_{xy}|$ we have $\ell_{d}(\gamma_{xy}) \leq c\delta_{\Omega}(x)$,
(v) $|\gamma| \cap \partial \Omega = \{a, b\}$,
(vi) $\gamma$ is a continuous $(c, c)$-quasigeodesic in $(\Omega, k_{\Omega})$: $\ell_{k_{\Omega}}(\gamma_{xy}) \leq c k_{\Omega}(x, y) + c$ for all $x, y \in |\gamma| \setminus \{a, b\}$.

The following is an analog of the anchor lemma 3.18 of [V1].

**Lemma 7.3.** Suppose $(X, d)$ is a $C_{a}$-annular convex geodesic space. If $0 < \lambda \leq 1/(2C_{a}^2)$, then every $\lambda$-arc point $x_{0} \in \Omega$ has a $c$-anchor with $c = c(\lambda, C_{a})$.

**Proof.** In this proof, $C$ and $C'$ denote constants that depend only on $\lambda$ and $C_{a}$, and their values may change from one occurrence to another as they represent all such constants occurring in this proof that we do not need to keep track of.

Let $a \in \partial \Omega$ such that $\delta_{\Omega}(x_{0}) = d(x_{0}, a)$. Since $x_{0}$ is a $\lambda$-arc point, there is a point $y \notin \Omega$ such that $\lambda \delta_{\Omega}(x_{0}) \leq d(a, y) < \delta_{\Omega}(x_{0})/\lambda$. Let $\gamma_{1}$ be a geodesic (with respect to the metric $d$) connecting $a$ to $x_{0}$. We break up the construction of the anchor into two cases.

**Case 1:** $d(a, y) \geq \delta_{\Omega}(x_{0})$. Then let $\beta_{0}$ be a geodesic with respect to the metric $d$ joining $y$ to $a$; $\beta_{0}$ intersects the sphere $S(a, \delta_{\Omega}(x_{0}))$ at exactly one point $w$. By the annular convexity of $X$ there is a rectifiable curve $\beta_{1}$ joining $x_{0}$ and $w$ in the annulus $B(a, C_{a}\delta_{\Omega}(x_{0})) \setminus B(a, \delta_{\Omega}(x_{0}))/C_{a}$ with $\ell_{d}(\beta_{1}) \leq C_{a}d(x_{0}, w)$. Let $\beta_{0}$ be the segment of $\beta_{0}$ with endpoints $y$ and $w$, and set $\gamma = \beta_{0} * \beta_{1} * \gamma_{1}$ to be the concatenation of the three paths $\beta_{0}, \beta_{1}$, and $\gamma_{1}$.

**Case 2:** $d(a, y) < \delta_{\Omega}(x_{0})$. In this case let $z \in |\gamma_{1}|$ be the unique point with $d(z, a) = d(a, y)$. Let $\beta_{0}$ be the subcurve of $\gamma_{1}$ between $x_{0}$ and $z$ oriented from $x_{0}$ to $z$, and by the annular convexity, let $\beta_{1}$ be a rectifiable curve in the annulus $B(a, C_{a}d(y, a)) \setminus B(a, d(y, a))/C_{a}$ joining $z$ and $y$ with $\ell_{d}(\beta_{1}) \leq C_{a}d(z, y)$. Now set $\gamma = \beta_{1} * \beta_{0} * \gamma_{1}$.

Once such a curve $\gamma$ has been constructed from the above cases, we modify this curve further. Since $y \notin \Omega$ and $\delta_{\Omega}(x_{0}) > \frac{\lambda}{3C_{a}}\delta_{\Omega}(x_{0})$, there is a point $x_{1} \in \Omega \cap (|\beta_{1}| \cup |\beta_{0}|)$ at

![Figure 1: The two cases in Lemma 7.3.](image-url)
which the curve $\beta_1 \cup \hat{\beta}_0$, beginning from the point $x_0$, first achieves $\delta_{\Omega}(x_1) = \frac{\lambda}{3C_a} \delta_{\Omega}(x_0)$. Let $\gamma_2 = (\beta_1 * \hat{\beta}_0)_{x_0 x_1}$. Then for all $x \in |\gamma_2| \setminus \{x_1\}$ we have

$$\delta_{\Omega}(x) > \frac{\lambda}{3C_a} \delta_{\Omega}(x_0).$$

(11)

Let $b \in \partial \Omega$ such that

$$\delta_{\Omega}(x_1) = d(x_1, b) = \frac{\lambda}{3C_a} \delta_{\Omega}(x_0).$$

(12)

Now the choice of $x_1$ implies that $x_1$ is the point on $\gamma_2$ nearest to $b$. Let $\beta_2$ be a $d$-geodesic from $x_1$ to $b$, and let $\gamma = \beta_2 * \gamma_2 * \gamma_1$.

We next verify that $\gamma$ satisfies conditions (i)–(vi) of the definition of a $c$-anchor. By construction $x_0 \in |\gamma|$, so condition (i) is satisfied. Condition (v) is also clear.

Note that by equation (12),

$$\ell_d(\gamma) = \ell_d(\gamma_1) + \ell_d(\gamma_2) \leq C_a d(x_0, w) + d(w, y) \leq 2C_a \delta_{\Omega}(x_0) + \frac{1}{\lambda} \delta_{\Omega}(x_0),$$

and hence

$$\ell_d(\gamma) \leq \left(1 + \frac{\lambda}{3C_a} + 2C_a + \frac{1}{\lambda} \right) \delta_{\Omega}(x_0).$$

(13)

In the situation of Case 1 above, we have

$$\ell_d(\gamma_2) \leq \ell_d(\beta_1) + \ell_d(\hat{\beta}_0) \leq C_a d(x_0, w) + d(w, y) \leq 2C_a \delta_{\Omega}(x_0) + \frac{1}{\lambda} \delta_{\Omega}(x_0),$$

and we obtain inequality (13) again. Since $\gamma_2$ does not intersect $B(a, \lambda \delta_{\Omega}(x_0)/C_a)$, and by equation (12) $\ell_d(\beta_2) = \lambda \delta_{\Omega}(x_0)/(3C_a)$, we see that $d(a, b) \geq \tau \delta_{\Omega}(x_0)$ — where

$$\tau = \frac{\lambda}{C_a} \left(1 - \frac{1}{3}\right).$$

Thus by inequality (13), $\ell_d(\gamma) \leq C d(a, b)$, and hence condition (ii) is also satisfied.

By Lemma 2.5, as $\gamma_{ax_0} = \gamma_1$, condition (iii) holds as well.

We now prove condition (iv). Recall that $\gamma_{xb} = \beta_2 * \gamma_2$. Again by Lemma 2.5, for all $x \in |\beta_2|$ we have $\ell_d(\gamma_{xb}) = \ell_d(\beta_2_{xb}) = \delta_{\Omega}(x)$. Note that by inequality (13), $\ell_d(\gamma_2) \leq C \delta_{\Omega}(x_0)$, and by the choice of $x_1$, $\delta_{\Omega}(z) \geq \lambda \delta_{\Omega}(x_0)/(3C_a)$ for $z \in |\gamma_2|$, see inequality (11). Thus,

$$\ell_{k_0}(\gamma_2) \leq \frac{3C_a}{\lambda \delta_{\Omega}(x_0)} \ell_d(\gamma_2) \leq C;$$

(14)

this will be useful in proving condition (vi). Let $x \in |\gamma_2|$. Then $\delta_{\Omega}(x) \geq \lambda \delta_{\Omega}(x_0)/(3C_a)$. Since $\ell_d(\gamma_{xb}) \leq \ell_d(\gamma) \leq C \delta_{\Omega}(x_0)$, we have $\ell_d(\gamma_{xb}) \leq C \delta_{\Omega}(x)$, and condition (iv) is satisfied.
It now only remains to prove condition (vi). We break this part of the proof up into cases again. Let \( x, x' \in |\gamma| \setminus \{ a, b \} \).

**Case (i):** \( x, x' \in |\gamma_1| \) or \( x, x' \in |\beta_2| \). By Lemma 2.5 and the proof of Proposition 2.6, both \( \gamma_1 \) and \( \beta_2 \) are geodesic rays in \((\Omega, k_\Omega)\). Hence we have \( \ell_{k_\Omega}(\gamma_{xx'}) = k_\Omega(x, x') \).

**Case (ii):** If both \( x, x' \) are in \( |\gamma_2| \), then by inequality (14),
\[
\ell_{k_\Omega}(\gamma_{xx'}) \leq \ell_{k_\Omega}(\gamma_2) \leq C \leq C + k_\Omega(x, x').
\] (15)

**Case (iii):** Suppose \( x \notin |\gamma_2| \) and \( x' \in |\gamma_2| \). If \( x \in |\gamma_1| \) and \( x' \in |\gamma_2| \), then by Case (i) and (15) (with \( x_0, x' \in |\gamma_2| \)),
\[
\ell_{k_\Omega}(\gamma_{xx'}) \leq \ell_{k_\Omega}(\gamma_{xx_0}) + \ell_{k_\Omega}(\gamma_2) \leq k_\Omega(x, x_0) + C \\
\leq k_\Omega(x, x') + k_\Omega(x', x_0) + C \\
\leq k_\Omega(x, x') + C.
\] (16)

Similarly, if \( x \in |\beta_2| \) and \( x' \in |\gamma_2| \) we have the estimate (16).

**Case (iv):** Finally, if \( x \in |\gamma_1| \) and \( x' \in |\beta_2| \), then
\[
\ell_{k_\Omega}(\gamma_{xx'}) = \ell_{k_\Omega}(\gamma_{xx_0}) + \ell_{k_\Omega}(\gamma_2) + \ell_{k_\Omega}(\gamma_{x_1x'}) \leq k_\Omega(x, x_0) + C + k_\Omega(x_1, x').
\] (17)

Let \( \tau' = \frac{\lambda}{6C_a} \).

**Subcase 1:** If \( d(a, x) \geq \tau' \delta_\Omega(x_0) \), then the proof of Proposition 2.6 shows that
\[
k_\Omega(x, x_0) \leq \log \left( \frac{d(a, x_0)}{d(a, x)} \right) \leq \log \left( \frac{1}{\tau'} \right) = C',
\]
and in this case, by inequality (17),
\[
\ell_{k_\Omega}(\gamma_{xx'}) \leq k_\Omega(x, x_0) + C + k_\Omega(x', x) + k_\Omega(x, x_0) + k_\Omega(x_0, x_1) \\
\leq C' + C + k_\Omega(x, x') + C' + C = k_\Omega(x, x') + C.
\] (18)

**Subcase 2:** \( d(a, x) < \tau' \delta_\Omega(x_0) \). By the choice of \( x_1 \), we have \( d(a, x_1) \geq \frac{\lambda}{C_a} \delta_\Omega(x_0) \). Therefore,
\[
d(b, a) \geq d(a, x_1) - d(x_1, b) \geq \frac{\lambda}{C_a} \delta_\Omega(x_0) - \frac{\lambda}{3C_a} \delta_\Omega(x_0) = \frac{2\lambda}{3C_a} \delta_\Omega(x_0).
\]

Hence for all \( z \in |\beta_2| \),
\[
d(z, a) \geq d(a, b) - d(z, b) \geq d(a, b) - d(x_1, b) \geq \frac{2\lambda}{3C_a} \delta_\Omega(x_0) - \frac{\lambda}{3C_a} \delta_\Omega(x_0) = \frac{\lambda}{3C_a} \delta_\Omega(x_0).
\]

In particular, the above estimate holds for \( x' \). Therefore by \( \ell_d(\gamma_{xx'}) \leq \ell_d(\gamma) \leq C \delta_\Omega(x_0) \),
\[
d(x, x') \geq d(x', a) - d(x, a) \geq \frac{\lambda}{3C_a} \delta_\Omega(x_0) - \tau' \delta_\Omega(x_0) = \frac{\lambda}{6C_a} \delta_\Omega(x_0) \geq C^{-1} \ell_d(\gamma_{xx'}).
\]
Since $\gamma$ satisfies conditions (iii) and (iv) of the definition of a $c$-anchor, so does $\gamma_{xx'}$. Now Lemma 2.3 implies

$$\ell_{k_{\Omega}}(\gamma_{xx'}) \leq C' \log \left(1 + \frac{\ell_d(\gamma_{xx'})}{\delta_{\Omega}(x) \wedge \delta_{\Omega}(x')}\right) \leq C' \log \left(1 + \frac{Cd(x, x')}{\delta_{\Omega}(x) \wedge \delta_{\Omega}(x')}\right)$$

$$\leq C' \log \left(1 + \frac{d(x, x')}{\delta_{\Omega}(x) \wedge \delta_{\Omega}(x')}\right) + C' \log C$$

$$\leq C'k_{\Omega}(x, x') + C' \log C. \quad (19)$$

This completes the proof. \hfill \Box

The following result is an analog of Theorem 2.4 of [V1], and provides a starlikeness condition for the space $(\Omega, k_{\Omega})$.

**Theorem 7.4.** Let $(X, d)$ be a $C_a$-annular convex proper geodesic space and $\Omega \subset X$ a rectifiably connected open subset with $\partial \Omega \neq \emptyset$. Set $\text{diam}'(\partial \Omega) = \text{diam}(\partial \Omega)$ if $\text{diam}(\Omega) < \infty$ and $\text{diam}'(\partial \Omega) = \infty$ if $\text{diam}(\Omega) = \infty$. If $(\Omega, k_{\Omega})$ is $\delta$-hyperbolic and $0 < \tau < 1$ such that $\text{diam}(\Omega) \leq \text{diam}'(\partial \Omega)/\tau$, then there is a constant $C = C(\delta, \tau, C_a)$ such that for all $a \in \partial^* \Omega$ and for all $x_0 \in \Omega$, there exists $b \in \partial^* \Omega$ and a quasihyperbolic geodesic line $\alpha : a \sim b$ such that $k_{\Omega}(x_0, |\alpha|) \leq C$.

**Proof.** Let $\lambda = \frac{\tau}{3C_a^2}$, and fix $a \in \partial^* \Omega$, $x_0 \in \Omega$. We divide the proof into two cases.

**Case 1:** $x_0$ is a $\lambda$-arc point. Then by Lemma 7.3, there is a $c$-anchor $\gamma$ with $x_0 \in |\gamma|$, where $c = c(\lambda, C_a) = c(\tau, C_a)$. Since $c$-anchors are $(c, c)$-quasigeodesics in $(\Omega, k_{\Omega})$, by Lemma 3.3 there is a quasihyperbolic geodesic line $\beta$ with endpoints $\xi, \eta \in \partial^* \Omega$ such that $HD_{k_{\Omega}}(|\beta|, |\gamma|) \leq M$, where $M = M(\delta, c, c) = M(\delta, \tau, C_a)$. Therefore there is a point $x_1 \in |\beta|$ such that $k_{\Omega}(x_1, x_0) \leq M$. If $a = \xi$ or $a = \eta$ then we are done. If $a \not\in \{\xi, \eta\}$, then let $\beta_1 : a \sim \xi$, $\beta_2 : a \sim \eta$ be two quasihyperbolic geodesic lines. Since geodesic triangles in $\Omega \cup \partial^* \Omega$ are $24\delta$-thin, we have $k_{\Omega}(x_1, |\beta_1| \cup |\beta_2|) \leq 24\delta$. Thus $k_{\Omega}(x_0, |\beta_i| \cup |\beta_j|) \leq M + 24\delta$, and hence $k_{\Omega}(x_0, |\beta_i|) \leq M + 24\delta$ for some $i \in \{1, 2\}$; we choose $\alpha = \beta_i$ for this particular $i$.

**Case 2:** $x_0$ is a $\lambda$-annulus point. Then there is a point $b \in \partial \Omega$ such that $\delta_{\Omega}(x_0) = d(x_0, b)$ and $B(b, \delta_{\Omega}(x_0)/\lambda) \setminus B(b, \lambda \delta_{\Omega}(x_0)) \subset \Omega$.

We now prove that there is a quasihyperbolic geodesic line $\beta$ intersecting the sphere $S(b, \delta_{\Omega}(x_0))$. If $\text{diam}(\Omega) = \infty$, pick $x_n, y_n \in \Omega$ with $d(x_0, x_n) \to \infty$ and $d(y_n, b) \to 0$; the sequence of geodesics $[x_n, y_n]$ in $(\Omega, k_{\Omega})$ has a subsequence converging to such a geodesic line. Now assume $\text{diam}(\Omega) < \infty$. Since $\text{diam}(\Omega) \leq \text{diam}(\partial \Omega)/\tau$, we have by the choice of $\lambda$,

$$\lambda \delta_{\Omega}(x_0) \leq \tau \delta_{\Omega}(x_0)/3 \leq \tau \text{diam}(\Omega)/3 \leq \text{diam}(\partial \Omega)/3;$$

hence there is a point $c \in \partial \Omega$ such that $d(c, b) \geq \lambda \delta_{\Omega}(x_0)$. The fact that the annulus $B(b, \delta_{\Omega}(x_0)/\lambda) \setminus B(b, \lambda \delta_{\Omega}(x_0)) \subset \Omega$ implies $c \not\in B(b, \delta(x_0)/\lambda)$. Select $x_n, y_n \in \Omega$ with $d(x_n, c) \to 0$ and $d(y_n, b) \to 0$; then by the Arzela-Ascoli theorem, the sequence of geodesics $[x_n, y_n]$ in $(\Omega, k)$ has a subsequence converging to a geodesic line intersecting the sphere $S(b, \delta_{\Omega}(x_0))$.

Given any $x, y \in S(b, \delta_{\Omega}(x_0))$, the annular convexity of $X$ implies that there is a path $\gamma : x \sim y$ with $\ell_d(\gamma) \leq C_d d(x, y)$ and $\gamma \subset B(b, C_a \delta_{\Omega}(x_0)) \setminus B(b, \delta_{\Omega}(x_0)/C_a)$. Since
that this abuse does not lead to inconsistencies.

Corollary 7.5. If \((\Omega, k_\Omega)\) is \(\delta\)-hyperbolic, \(\text{diam}(\partial\Omega) > 0\), and \(0 < \alpha < 1\), then for all \(a \in \partial^*\Omega\) and all \(x_0 \in \Omega\),

- if \(x_0\) is a \(\alpha\)\(\frac{\alpha}{3C_a^2}\)-arc point, or
- if \(x_0\) is a \(\alpha\)\(\frac{\alpha}{3C_a^2}\)-annulus point with \(\delta_\Omega(x_0) \leq \frac{1}{\alpha}\text{diam}(\partial\Omega)\),

then there is a point \(b \in \partial^*\Omega\) and a quasihyperbolic geodesic line \(\gamma : a \sim b\) such that \(k_\Omega(x_0, |\alpha|) \leq C\) with \(C = C(\delta, \alpha, C_a)\).

The following result follows from the fact that triangles in \(\Omega \cup \partial^*\Omega\) are 24\(\delta\)-thin.

Lemma 7.6 (Lemma 6.35 of [V3]). If \((\Omega, k_\Omega)\) is \(\delta\)-hyperbolic and is roughly starlike with constant \(C_0\) with respect to \(a \in \partial^*\Omega\), then whenever \(x_1, x_2 \in \Omega\) there is a quasihyperbolic geodesic line \(\alpha : a_1 \sim a_2\) for some \(a_1, a_2 \in \partial^*\Omega\) such that \(k_\Omega(x_i, |\alpha|) \leq C = C(C_0, \delta)\) for \(i = 1, 2\).

### 8 A “carrot” lemma for quasihyperbolic geodesics

In this section we show that, under the assumptions of Theorem 9.1, quasigeodesic lines in \(\Omega\) have properties very similar to the conditions for uniform curves. The proof of Theorem 9.1 will essentially be reduced to this situation.

As in the previous sections, \((X, d)\) is a \(C_a\)-annular convex proper geodesic space, and \(\Omega \subset X\) is an unbounded rectifiably connected open subset with \(\partial\Omega \neq \emptyset\). We suppose \((\Omega, k_\Omega)\) is \(\delta\)-hyperbolic, there is a natural map \(\phi : (\partial^*\Omega, d_{w,\epsilon}) \to (\partial\Omega, d)\) (for some \(w \in \Omega\) and \(0 < \epsilon \leq \epsilon_0(\delta)\)) and that \(\phi\) is \(\eta\)-quasimöbius for some \(\eta\). Recall that \(\partial^*\Omega = \partial\Omega \cup \{\infty\}\) and that the cross ratio in \((\partial^*\Omega, d)\) is well-defined; see the second paragraph of Section 6. By Corollary 5.4, we may assume that for each \(x \in \Omega\), \(\phi : (\partial^*\Omega, d_{x,\epsilon}) \to (\partial\Omega, d)\) is \(\eta\)-quasimöbius. By an abuse of notation, for any \(\xi \in \partial^*\Omega\), we denote \(\phi(\xi)\) also as \(\xi\), and for any \(\xi \in \partial\Omega\), denote \(\phi^{-1}(\xi)\) also by \(\xi\); Theorem 4.6 (1) together with the discussion in Section 6 shows that this abuse does not lead to inconsistencies.

The following is a simplified version of the distance carrot lemma 3.36 of [V1].

**Lemma 8.1.** If \(\alpha\) is a quasihyperbolic geodesic line with endpoints \(b, \infty \in \partial\Omega\), then for all \(x \in |\alpha|\) we have

\[
d(x, b) \leq C \delta_\Omega(x),
\]

where \(C = C(\delta, C_a, \eta)\).
Proof. Let $x \in |\alpha|$ and $\lambda = \frac{1}{32C^2a} e^{-4c^3}$. As before, we break the proof up into two cases.

**Case 1:** $x$ is a $\lambda$-annulus point. Then there is a point $a \in \partial \Omega$ such that $\delta_{\Omega}(x) = d(x,a)$ and $B(a, \delta_{\Omega}(x)/\lambda) \setminus B(a, \lambda \delta_{\Omega}(x)) \subseteq \Omega$. Since $\alpha : b \sim \infty$, we can find a sequence $(v_n)$ from $|\alpha|$ with $v_n = \alpha(t_n), t_n \to \infty$, such that $v_n \not\in B(a, \delta_{\Omega}(x)/\lambda)$ for all $n$. If $b \not\in B(a, \lambda \delta_{\Omega}(x))$, then $b \not\in B(a, \delta_{\Omega}(x)/\lambda)$, and hence by Lemma 2.3 applied to the quasi-hyperbolic geodesic $\alpha_{\delta_{\Omega}}$, with $t = 2e^{4c^3}\delta_{\Omega}(x)$, we see that $d(x,a) \geq 2\delta_{\Omega}(x)$, which violates the fact that $\delta_{\Omega}(x) = d(x,a)$. Hence $b \in B(a, \lambda \delta_{\Omega}(x))$. Therefore,

$$d(x,b) \leq d(x,a) + d(a,b) \leq \delta_{\Omega}(x) + \lambda \delta_{\Omega}(x) \leq 2\delta_{\Omega}(x).$$

**Case 2:** $x$ is a $\lambda$-arc point. Then by Lemma 7.3 there is a $c$-anchor $\tau : a_1 \sim a_2$ with $a_1, a_2 \in \partial \Omega$ and $x \in |\tau|$, where $c = c(\lambda, C_a) = c(C_a)$. Let $\beta : a_1 \sim a_2$ be a quasi-hyperbolic geodesic line. Since $\tau$ is a $(c, c)$-quasigeodesic in $(\Omega, k_{\Omega})$, we have

$$k_{\Omega}(|\alpha|, |\beta|) \leq k_{\Omega}(|\alpha|, |\tau|) + k_{\Omega}(|\tau|, |\beta|) \leq 0 + C(\delta, c, c) = C(\delta, C_a).$$

Set $Q = (a_1, \infty, b, a_2)$. Then $sd(Q) \leq k_{\Omega}(|\alpha|, |\beta|) \leq C$. Since $\epsilon \leq c_0(\delta) \leq 1$, by Corollary 5.2 we have $cr(Q, d_{x,\epsilon}) \leq c_0 c^{\epsilon sd(Q)} \leq c_0 e^{cC} \leq c_0 e^{C} = C$, where $c_0 = 4e^{86}$.

Therefore by the quasimöbius property of the natural map $\phi$,

$$\frac{d(a_1, b)}{d(a_1, a_2)} \leq \eta(cr(Q, d_{x,\epsilon})) \leq \eta(C) = C(\delta, C_a, \eta);$$

that is, $d(a_1, b) \leq Cd(a_1, a_2)$. Since $\tau$ is a $c$-anchor of $x$ with endpoints $a_1$ and $a_2$, by properties (iii) and (iv) of the Definition 7.2 of anchors (with $x_0 = x$ here),

$$\ell_d(\tau_{a_1 x}) \leq C \delta_{\Omega}(x) \quad \text{and} \quad \ell_d(\tau_{x a_2}) \leq C \delta_{\Omega}(x),$$

and therefore $d(a_1, a_2) \leq \ell_d(\tau) \leq 2C \delta_{\Omega}(x)$; hence $d(a_1, b) \leq C \delta_{\Omega}(x)$. Finally,

$$d(x,b) \leq d(x,a_1) + d(a_1, b) \leq d(x,a_1) + C \delta_{\Omega}(x) \leq \ell_d(\tau_{x a_1}) + C \delta_{\Omega}(x) \leq C \delta_{\Omega}(x),$$

where we used property (iii) of Definition 7.2 again. This concludes the proof.

□

**Lemma 8.2.** Let $x_0 \in \Omega$ and $\tau : a_1 \sim a_2$ be a $c$-anchor for $x_0$ (for some $a_1, a_2 \in \partial \Omega$). Let $\alpha : a_1 \sim a_2$ and $\alpha_i : a_i \sim \infty$ $(i = 1, 2)$ be quasi-hyperbolic geodesic lines. Let $x \in |\alpha|$ be such that $k_{\Omega}(x, |\alpha|) \leq 24 \delta$ for $i = 1, 2$. Then $k_{\Omega}(x, x_0) \leq c' = c'(\delta, c, C_a, \eta)$.

**Proof.** Since $\tau$ is a $(c, c)$-quasigeodesic and $\tau$ and $\alpha$ have the same endpoints, we have the control $HD_{k_{\Omega}}(|\tau|, |\alpha|) \leq c_1 = c_1(\delta, c)$. Fix $y \in |\tau|$ with $k_{\Omega}(x, y) \leq c_1$. We claim that there is a constant $c_2 = c_2(\delta, c, C_a, \eta)$ such that $\delta_{\Omega}(y) \geq \delta_{\Omega}(x_0)/c_2$.

For now assuming the claim to hold, we proceed as follows. Since the restriction $\tau$ to the subcurve $\tau_{x_0 y}$ satisfies the assumptions of Lemma 2.3 (this is because $\tau$ is an anchor), Lemma 2.3 implies $k_{\Omega}(x_0, y) \leq c_3 = c_3(\delta, c, C_a, \eta)$, and hence

$$k_{\Omega}(x, x_0) \leq k_{\Omega}(x, y) + k_{\Omega}(y, x_0) \leq c_3 + c_1.$$
It now only remains to prove the claim. To this end, let \(c_2 = 2c[c - 1 + (C + 1)e^{c_1 + 24\delta}]\), where \(C = C(\delta, C_\alpha, \eta)\) is the constant from Lemma 8.1. Suppose \(\delta_\Omega(y) < \delta_\Omega(x_0)/c_2\). We may assume \(y \in |\tau_{a_2x_0}|\). Then condition (iv) of a \(c\)-anchor implies

\[
d(a_2, y) \leq \ell_d(\tau_{a_2y}) \leq c\delta_\Omega(y) \leq c\delta_\Omega(x_0)/c_2.
\]

Let \(y_1 \in |\alpha_1|\) with \(k_\Omega(y_1, x) \leq 24\delta\). Then \(k_\Omega(y_1, y) \leq c_1 + 24\delta\). Therefore, from Lemma 2.2,

\[
d(y_1, y) \leq (e^{c_1 + 24\delta} - 1)\delta_\Omega(y) \leq \frac{e^{c_1 + 24\delta} - 1}{c_2} \delta_\Omega(x_0),
\]

and hence

\[
\delta_\Omega(y_1) \leq \delta_\Omega(y) + d(y, y_1) \leq \frac{e^{c_1 + 24\delta} - 1}{c_2} \delta_\Omega(x_0).
\]

On the other hand, Lemma 8.1 applied to \(\alpha_1\) and \(y_1\) implies that

\[
d(a_1, y_1) \leq C\delta_\Omega(y_1) \leq \frac{Ce^{c_1 + 24\delta}}{c_2} \delta_\Omega(x_0).
\]

Now the triangle inequality implies

\[
d(a_1, a_2) \leq d(a_1, y_1) + d(y_1, y) + d(y, a_2) \leq \frac{\delta\Omega(x_0)}{2c} < \frac{\delta\Omega(x_0)}{c}.
\]

This is impossible since by condition (ii) of a \(c\)-anchor,

\[
d(a_1, a_2) \geq \ell_d(\tau)/c \geq d(x_0, a_1)/c \geq \delta\Omega(x_0)/c.
\]

The following is the analog in our setting of the length carrot lemma 3.40 of [V1].

**Lemma 8.3.** If \(\alpha\) is a quasihyperbolic geodesic line with endpoints \(b, \infty \in \partial\Omega\), then there is a constant \(C = C(\delta, C_\alpha, \eta, \varepsilon)\) such that for all \(x \in |\alpha|\),

\[
\ell_d(\alpha_{bx}) \leq C\delta\Omega(x).
\]

**Proof.** Let \(\alpha : \mathbb{R} \to \Omega\) be the \(k\)-arclength parametrization of \(\alpha\) with \(\lim_{t \to -\infty} \alpha(t) = b\) and \(\lim_{t \to \infty} \alpha(t) = \infty\). For each \(n \in \mathbb{Z}\) let

\[
t_n = \sup\{t \in \mathbb{R} : \delta\Omega(\alpha(t)) \leq 2^n\}.
\]

Since \(\lim_{t \to \infty} \alpha(t) = \infty\), Lemma 8.1 implies that for each \(n\) we have \(t_n < \infty\), \(\delta\Omega(\alpha(t_n)) = 2^n\), and \(t_n < t_{n+1}\).

Fix \(x \in |\alpha|\). Then there exists \(n \in \mathbb{Z}\) for which \(x \in |\alpha|_{(t_n, t_{n+1})}\). We have

\[
\ell_d(\alpha_{bx}) \leq \ell_d(\alpha_{|(-\infty, t_{n+1})|}) = \sum_{j=-\infty}^{n} \ell_d(\alpha_{|t_j, t_{j+1}|}).
\]

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By Lemma 2.2,
\[ \ell_d(\alpha_{|t_j,t_{j+1}}) \leq \delta_\Omega(\alpha(t_j)) \left[ e^{L_\Omega(\alpha_{|t_j,t_{j+1}})} - 1 \right]. \]

Hence,
\[ \ell_d(\alpha_{bx}) \leq \sum_{j=-\infty}^{n} 2^j \left[ e^{L_\Omega(\alpha_{|t_j,t_{j+1}})} - 1 \right]. \]

It suffices to show that there is a constant \( K = K(\delta, \eta, C_a, \varepsilon) \) such that for all \( j \in \mathbb{Z} \),
\[ \ell_{k_\Omega}(\alpha_{|t_j,t_{j+1}}) \leq K, \]
for then,
\[ \ell_d(\alpha_{bx}) \leq (e^K - 1) \sum_{j=-\infty}^{n} 2^j \leq e^K 2^n \sum_{j=-\infty}^{0} 2^j = 2e^K 2^n. \]

On the other hand, as \( x \in |\alpha_{|t_n,t_{n+1}}| \), we have \( \delta_\Omega(x) \geq 2^n \). Thus we can infer that
\[ \ell_d(\alpha_{bx}) \leq 2e^K \delta_\Omega(x), \]
concluding the proof of the lemma.

It therefore now remains to prove inequality (20).

Let \( \lambda = \frac{1}{40C^3e^{4C^3}} \), where \( C = \max\{C_a, 2C\} \) with \( C \) the constant from the conclusion of Lemma 8.1.

**Case 1:** Both \( x_1 := \alpha(t_n) \) and \( x_2 := \alpha(t_{n+1}) \) are \( \lambda \)-arc points. By the choice of \( t_{n+1} \), we know that \( \delta_\Omega(x_2) = 2^{n+1} \) and so \( \delta_\Omega(x_2) = 2 \cdot 2^n \leq 2\delta_\Omega(x) \) for every \( x \in |\alpha_{|t_n,t_{n+1}}| \). By Lemma 7.3 there are \( c \)-anchors \( \tau_i : a_i \sim c_i \) for \( i = 1, 2 \), with \( x_i \in |\tau_i| \) and \( c = c(\lambda, C_a) \). Without loss of generality, we may assume that \( d(a_i, b) \geq d(c_i, b) \) for \( i = 1, 2 \).

Fix \( y_0 = \alpha(t_0) \) with \( t_0 \) sufficiently large. For \( i = 1, 2 \), let \( \alpha_i \) be a geodesic ray connecting \( y_0 \) to \( a_i \) and \( \beta_i \) a geodesic ray from \( y_0 \) to \( c_i \). Set \( X = |\alpha_{y_0a}| \cup |\alpha_{yc}| \cup |\alpha| \cup |\alpha_2| \cup |\beta_1| \cup |\beta_2| \). By Theorem 3.4 there is a tree \( T(X) \) and a map \( u : X \rightarrow T(X) \) with the properties stated in Theorem 3.4. We denote the metric on \( T(X) \) by \( d_T \). Let \( a'_i \in \partial T(X) \) be such that \( u \) is an isometry from \( |\alpha_i| \) onto the geodesic \( [u(y_0), a'_i] \) in \( T(X) \). We similarly define \( c'_i, \infty', b' \in \partial T(X) \).

Let \( y'_i \in T(X) \) be the branch point of \( [u(y_0), a'_i] \) and \( [u(y_0), c'_i] \), that is, \( [u(y_0), a'_i] \cap [u(y_0), c'_i] = [u(y_0), y'_i] \). Choose \( y_{ia} \in |\alpha_i| \) and \( y_{ic} \in |\beta_i| \) with \( u(y_{ia}) = u(y_{ic}) = y'_i \). Then \( k_\Omega(y_{ia}, y_{ic}) \leq c(\delta) \). Fix a geodesic \( \gamma_i \) joining \( a_i \) and \( c_i \). Then the argument in the proof of Lemma 5.1 shows that
\[ HD_{k_\Omega}(|\gamma_i|, |\alpha_i|y_{ia}, a_i| \cup |\beta_i|y_{ic}, c_i|) \leq c(\delta). \]

Pick \( y_i \in |\gamma_i| \) with \( k_\Omega(y_i, y_{ia}) \leq c(\delta) \). Then \( k_\Omega(y_i, y_{ic}) \leq 2c(\delta) \). The proof of Lemma 8.2 shows that \( k_\Omega(y_i, x_i) \leq c_3 = c_3(\delta, C_a, \eta) \). Hence \( k_\Omega(x_i, y_{ia}) \leq c_3 + c(\delta) \). Set \( C_1 = c_3 + c(\delta) \). We have \( d_T(u(x_i), y'_i) \leq k_\Omega(y_i, x_i) \leq C_1 \).

Consider the following subtrees of \( T(X) \): \( Y'_i = [\cdot', c'_i] \cup [\cdot', a'_i], Z'_i = [y'_i, c'_i] \cup [y'_i, a'_i] \).

Notice that \( Y'_i \) is tripod-shaped and that \( Z'_i \) is a geodesic line. Let \( z'_i \in Y'_i \) be the point where \( [\cdot', b'] \) branches off from \( Y'_i \): \( [\cdot', b'] \cap Y'_i = [z'_i, \cdot'] \). Since \( u(x_i) \in u(|\alpha|) = [\cdot', b'] \)
and \( y'_i \in Z'_i \), the inequality \( d_T(u(x_i), y'_i) \leq C_1 \) implies that \( d_T([\infty', b'], Z'_i) \leq C_1 \). It follows that the branch point \( z'_i \) has to be close to \( Z'_i \), specifically, \( d_T(z'_i, Z'_i) \leq C_1 \).

Let \( Q_i = (a_i, \infty, c_i, b) \) for \( i = 1, 2 \); then

\[
cr(Q_i, d) = \frac{d(a_i, c_i)}{d(a_i, b)} \leq \frac{d(a_i, b) + d(c_i, b)}{d(a_i, b)} \leq 2 \frac{d(a_i, b)}{d(a_i, b)} = 2.
\]

Since \( \phi : (\partial^* \Omega, d_{w, \epsilon}) \to (\partial \Omega, d) \) is \( \eta \)-quasimöbius, we have that \( \phi^{-1} : (\partial \Omega, d) \to (\partial^* \Omega, d_{w, \epsilon}) \) is \( \eta' \)-quasimöbius with \( \eta'(t) = \frac{1}{\eta(t)^{-1}(1/t)} \). Therefore,

\[
cr(Q_i, d_{w, \epsilon}) \leq \eta' (cr(Q_i, d)) \leq \eta'(2) = C = C(\eta).
\]

By Corollary 5.3 we have \( k_\Omega([a_i, b], [c_i, \infty]) \leq c_4 = c_4(\eta, \varepsilon, \delta) \) for any geodesic \([a_i, b]\) joining \( a_i \) and \( b \) and any geodesic \([c_i, \infty]\) joining \( c_i \) and \( \infty \). Now the property of \( u \) implies \( d_T([a'_i, b'], [c'_i, \infty']) \leq c_4 + c(\delta) \). Let \( w'_i \in [\infty', b'] \) be the branch point of \([\infty', b']\) and \([\infty', a'_i]\); \([\infty', b'] \cap [\infty', a'_i] = [w'_i, \infty'] \). If \( w'_i \in [y'_i, \infty]\{y'_i\} \), then \( w'_i = z'_i \) and hence \( d_T(y'_i, w'_i) = d_T(Z'_i, z'_i) \leq C_1 \); on the other hand, if \( w'_i \notin [y'_i, a'_i] \), then the inequality \( d_T([a'_i, b'], [c'_i, \infty']) \leq c_4 + c(\delta) \) implies that the branch point \( w'_i \) has to be close to \( y'_i \), that is,

\[
d_T(u'_i, y'_i) = d_T([a'_i, b'], [c'_i, \infty']) \leq c_4 + c(\delta). \]

In either case, we have \( d_T(y'_i, w'_i) \leq C_2 := \max\{C_1, c_4 + c(\delta)\} \). It follows that

\[
d_T(u(x_i), w'_i) \leq C_1 + C_2.
\]

As \( \tau_i \) is a \( c \)-anchor of \( x_i \), by property (iii) of Definition 7.2,

\[
d(a_i, x_i) \leq \ell_d(\tau_i |_{a_i, x_i}) \leq c \delta_{\Omega}(x_i),
\]

and by property (ii) of this definition,

\[
\delta_{\Omega}(x_i) \leq d(x_i, a_i) \leq \ell_d(\tau_i) \leq c d(a_i, c_i) \leq 2 c d(a_i, b).
\]

By Lemma 8.1, \( d(x_i, b) \leq C \delta_{\Omega}(x_i) \). From the above group of inequalities,

\[
\delta_{\Omega}(x_2) = 2 \delta_{\Omega}(x_1) \leq 4 c d(a_1, b),
\]

and hence

\[
d(a_2, b) \leq d(a_2, x_2) + d(x_2, b) \leq c \delta_{\Omega}(x_2) + C \delta_{\Omega}(x_2) \leq 4c(c + C) d(a_1, b).
\]

Thus, considering the quadruple \( Q_3 = (b, \infty, a_2, a_1) \), we obtain

\[
cr(Q_3, d) = \frac{d(a_2, b)}{d(a_1, b)} \leq 4c(c + C) = C = C(\delta, C_a, \eta).
\]
It follows that $cr(Q_3, d_{w,\epsilon}) \leq \eta'(C)$. By Corollary 5.3 we have
\[ k_\Omega([b, a_1], [a_2, \infty]) \leq c_5 = c_5(\delta, C_a, \eta, \varepsilon) \]
for any geodesic $[b, a_1]$ connecting $b$ to $a_1$ and any geodesic $[a_2, \infty]$ connecting $a_2$ to $\infty$. Now the property of $u$ implies $d_T([b', a'_1], [a'_2, \infty']) \leq C_3 := c_5(\delta)$.

If $k_\Omega(x_1, x_2) \leq 3C_1 + 3C_2 + C_3$, then we are done. If $k_\Omega(x_1, x_2) \geq 3C_1 + 3C_2 + C_3$ then $d_T(u(x_1), u(x_2)) = k(x_1, x_2) \geq 3C_1 + 3C_2 + C_3$. Then, since we have $w'_i \in [b', \infty']$, $d_T(w'_i, u(x_i)) \leq C_1 + C_2$, and $u(x_1) \in [u(x_2), b']$, we have $w'_1 \in [w'_2, b']$ and $d_T(w'_1, w'_2) \geq C_1 + C_2 + C_3$. It follows that
\[ d_T([b', a'_1], [a'_2, \infty']) = d_T(w'_1, w'_2) \geq C_1 + C_2 + C_3 > C_3, \]
contradicting the inequality $d_T([b', a'_1], [a'_2, \infty']) \leq C_3$ from the preceding paragraph.

**Case 2:** At least one of $x_1 = \alpha(t_n), x_2 = \alpha(t_{n+1})$ is a $\lambda$-annulus point. Since $\lambda = \frac{40C^3}{400C^3}$ with $C$ at least as large as the constant in the conclusion of Lemma 8.1, and as $x_i$ is a $\lambda$-annulus point for some $i \in \{1, 2\}$, there exists $a \in \partial \Omega$ such that $\delta_\Omega(x_i) = 2^{n-1+i} = d(a, x_i)$ and $B(a, \delta_\Omega(x_i)/\lambda) \cup B(a, \lambda \delta_\Omega(x_i)) \subset \Omega$. Thus we see that
\[ d(a, b) \leq d(b, x_i) + d(x_i, a) \leq (C + 1) \delta_\Omega(x_i) \leq 2C \delta_\Omega(x_i) \leq \frac{1}{\lambda} \delta_\Omega(x_i). \]

We break the rest of the proof up into two subcases.

**Subcase 2(a):** We consider the case when $x_2$ is a $\lambda$-annulus point. As $\delta_\Omega(x_1) = 2^n$,
\[ d(x_1, a) \geq \delta_\Omega(x_1) = \frac{1}{2} \delta_\Omega(x_2) > 2C_\lambda \delta_\Omega(x_2). \]

Therefore, $x_1 \in \Omega \setminus B(a, \delta_\Omega(x_2)/2) \subset \Omega \setminus B(a, \lambda \delta_\Omega(x_2))$. However, $\delta_\Omega(x_1) = 2^n \geq d(x_1, b)/C$, and hence
\[ d(x_1, a) \leq d(x_1, b) + d(b, a) \leq C \delta_\Omega(x_2) + (C + 1) \delta_\Omega(x_2) = (2C + 1) \delta_\Omega(x_2). \]

As
\[ \frac{1}{\lambda} = 40C^3 e^{4C^3} > 2(2C + 1)C_a, \]
we have
\[ d(x_1, a) < \frac{1}{2C_\lambda} \delta_\Omega(x_2), \]
that is, $x_1 \in B(a, \delta_\Omega(x_2)/(2C_a)) \setminus B(a, 2C_a \lambda \delta_\Omega(x_2))$. Hence by the annular convexity of $X$, there is a curve $\beta$ joining $x_1$ and $x_2$ in $B(a, \delta_\Omega(x_2)/(2\lambda)) \setminus B(a, 2\lambda \delta_\Omega(x_2)) \subset \Omega$ with
\[ \ell_d(\beta) \leq C_a d(x_1, x_2) \leq C_a d(x_1, b) + d(x_2, b) \leq C_a \left[ C \delta_\Omega(x_1) + C \delta_\Omega(x_2) \right] \leq 3C^2 \delta_\Omega(x_1). \]

For all $w \in |\beta|$, we have
\[ \delta_\Omega(w) \geq \min \left\{ \left( \frac{1}{\lambda} - \frac{1}{2\lambda} \right) \delta_\Omega(x_2), (2\lambda - \lambda) \delta_\Omega(x_2) \right\} = \lambda \delta_\Omega(x_2). \]
Thus,
\[ k_\Omega(x_1, x_2) \leq \ell_{k_\Omega}(\beta) = \int_\beta \frac{1}{\delta_\Omega(z)} dz \leq \frac{1}{\lambda \delta_\Omega(x_2)} \ell_d(\beta) = \frac{3C^2 \delta_\Omega(x_1)}{\lambda \delta_\Omega(x_2)} = \frac{3C^2}{2\lambda}, \]
proving inequality (20) in this subcase.

**Subcase 2(b):** \( x_1 \) is a \( \lambda \)-annulus point. The proof of this subcase is similar to the proof of Subcase 2(a), and is left to the reader.

**Lemma 8.4.** Let \( a_1, a_2 \in \partial \Omega \) and \( \alpha : a_1 \sim a_2 \) be a quasihyperbolic geodesic line. Then for all \( z \in |\alpha|, \)
\[ \delta_\Omega(z) \leq K d(a_1, a_2) \]
where \( K = K(\delta, C_\alpha, \eta) \) is independent of \( a_1, a_2, \alpha. \)

**Proof.** Let \( z \in |\alpha| \) and \( \lambda = \frac{e^{-4C_\alpha^2}}{65C_\alpha^2} \). Two possibilities arise.

**Case 1:** \( z \) is a \( \lambda \)-annulus point. Then there exists \( a \in \partial \Omega \) such that \( \delta_\Omega(z) = d(z, a) \)
and \( B(a, \delta_\Omega(z)/\lambda) \setminus B(a, \lambda \delta_\Omega(z)) \subset \Omega \). Lemma 2.4 implies that exactly one of \( a_1, a_2 \) lies in
\( B(a, \lambda \delta_\Omega(z)) \) with the other one in \( X \setminus B(a, \delta_\Omega(z)/\lambda). \) Hence we obtain the desired conclusion
\[ d(a_1, a_2) \geq |d(a_2, a) - d(a_1, a)| \geq \left( \frac{1}{\lambda} - \lambda \right) \delta_\Omega(z). \]

**Case 2:** \( z \) is a \( \lambda \)-arc point. Then by Lemma 7.3 there is a \( c \)-anchor \( \tau : b_1 \sim b_2 \) for \( z \)
with \( c = c(\lambda, C_\alpha) = c(C_\alpha) \). Let \( \beta : b_1 \sim b_2 \) and \( \alpha_i : b_i \sim \infty \) \( (i = 1, 2) \) be quasihyperbolic geodesic lines. Let \( x \in |\beta| \) be such that \( k_\Omega(x, |\alpha_i|) \leq 24\delta \) for \( i = 1, 2 \). By Lemma 8.2,
\[ k_\Omega(x, z) \leq c' = c'(\delta, C_\alpha, \eta) = c'(\delta, C_\alpha, \eta). \]
It follows that
\[ k_\Omega(|\alpha|, |\alpha_i|) \leq k_\Omega(z, |\alpha_i|) \leq k_\Omega(z, x) + k_\Omega(x, |\alpha_i|) \leq c' + 24\delta. \]
For \( i = 1, 2 \), set \( P_i = (a_i, \infty, b_i, a_2). \) Corollary 5.3 implies that
\[ cr(P_i, d_{w,\epsilon}) \leq C = C(c' + 24\delta) = C(\delta, C_\alpha, \eta). \]
Since the natural map is \( \eta \)-quasimöbius, we have
\[ cr(P_i, d) = \frac{d(a_1, b_i)}{d(a_1, a_2)} \leq \eta(C). \]
By the definition of a \( c \)-anchor, we have
\[ \delta_\Omega(z) \leq d(b_1, z) \leq \ell_d(\tau) \leq c d(b_1, b_2) \leq c [d(b_1, a_1) + d(a_1, b_2)] \leq 2c \eta(C) d(a_1, a_2), \]
which is the desired estimate. \( \square \)

The following lemma is the analog of Lemma 8.3 for quasihyperbolic geodesic lines that do not have \( \infty \) as one of the endpoints, and says that there is a “banana”-shaped region with respect to the metric \( d \) around such a line in \( \Omega \). The proof in [V1] holds in our case, and we skip the details here.
Lemma 8.5 (Lemma 3.54 of [V1]). Suppose \( a_1, a_2 \in \partial \Omega \) and \( \alpha : a_1 \sim a_2 \) is a quasihyperbolic geodesic line.

(i) There exists \( \xi_\alpha \in |\alpha| \) such that if \( x_1, x_2 \in |\alpha| \) with \( k_\Omega(x_2, \xi_\alpha) \leq k_\Omega(x_1, \xi_\alpha) \) or if \( x_1, x_2 \in |\alpha| \) with \( k_\Omega(x_2, \xi_\alpha) \leq k_\Omega(x_1, \xi_\alpha) \), then \( \ell_d(\alpha_{x_1x_2}) \leq C \delta_\Omega(x_2) \) for some \( C = C(\delta, \eta, C_a, \varepsilon) \).

(ii) If \( y_1, y_2 \in |\alpha| \) with \( \max\{\delta_\Omega(y_1), \delta_\Omega(y_2)\} \leq 2d(y_1, y_2) \), then \( \ell_d(\alpha_{y_1y_2}) \leq C d(y_1, y_2) \), where \( C = C(\delta, \eta, C_a, \varepsilon) \).

9 Sufficiency

In this section we prove the main result of the paper. This result (Theorem 9.1), together with Theorem 6.2, provides a characterization of uniform domains among Gromov hyperbolic domains in annular convex metric spaces in terms of the quasiconformal structure on the Gromov boundary.

Theorem 9.1. Let \((X, d)\) be a c-quasiconvex and c-annular convex proper metric space, and \( \Omega \subset X \) a rectifiably connected open subset with \( \partial \Omega \neq \emptyset \). Suppose that \((\Omega, k_\Omega)\) is \( \delta \)-hyperbolic and that the natural map \( \phi : (\partial^* \Omega, d_{\omega, \varepsilon}) \to (\partial' \Omega, d) \) exists and is \( \eta \)-quasimöbius. Then \((\Omega d)\) is \( c_1 \)-uniform with \( c_1 = c_1(c, \delta, \eta, \varepsilon) \).

The following lemma reduces Theorem 9.1 to the case of geodesic metric spaces.

Lemma 9.2. Let \((X, d)\) be a proper c-quasiconvex metric space, and \( \Omega \subset X \) a rectifiably connected open subset with \( \partial \Omega \neq \emptyset \). Let \( d' \) be the length metric on \( X \) associated with \( d \), and \( k'_\Omega \) the quasihyperbolic metric on \( \Omega \subset (X, d') \).

(i) For all \( x, y \in X \), we have \( d(x, y) \leq d'(x, y) \leq c d(x, y) \); in particular, \((X, d')\) is a proper geodesic space;

(ii) If \((X, d)\) is \( C_a \)-annular convex, then \((X, d')\) is \( c' \)-annular convex with \( c' = c'(c, C_a) \);

(iii) For all \( x, y \in \Omega \), we have \( k_\Omega(x, y) / c \leq k'_\Omega(x, y) \leq c k_\Omega(x, y) \);

(iv) If \((\Omega, d')\) is \( c' \)-quasiconvex, then \((\Omega, d)\) is \( c'' \)-uniform with \( c'' = c' c \);

(v) If \((\Omega, k_\Omega)\) is \( \delta \)-hyperbolic, then \((\Omega, k'_\Omega)\) is \( \delta' \)-hyperbolic with \( \delta' = \delta'(\delta, c) \);

(vi) Suppose \((\Omega, k_\Omega)\) is \( \delta \)-hyperbolic and that there is a natural map \( \phi : (\partial^* \Omega, d_{\omega, \varepsilon}) \to (\partial' \Omega, d) \) for some \( x \in \Omega \) and \( 0 < \varepsilon \leq \varepsilon_0(\delta) \) and \( \phi \) is \( \eta \)-quasimöbius. Then, for \( 0 < \varepsilon' \leq \varepsilon_0(\delta') \) there is a natural map \( \phi' : (\partial^* \Omega, d_{\omega, \varepsilon'}) \to (\partial' \Omega, d') \) such that \( \phi' \) is \( \eta' \)-quasimöbius with \( \eta' = \eta'(\eta, \delta, c, \varepsilon, \varepsilon') \). Here \( \delta' \) is the constant from (v).

Proof. (i): For any \( x, y \in X \), there is a path \( \gamma : x \sim y \) with \( \ell(\gamma) \leq c d(x, y) \). Hence \( d'(x, y) \leq c d(x, y) \). The inequality \( d(x, y) \leq d'(x, y) \) is clear. Since \((X, d)\) is proper, it now follows that \((X, d')\) is also proper. Being a proper length space, \((X, d')\) has to be geodesic.

(ii): This follows easily from (i) and the annular convexity of \((X, d)\).
For any $x \in \Omega$, let $\delta'_\Omega(x) = d'(x, \partial \Omega)$. It can be verified that $\ell_d(\gamma) \leq \ell_{d'}(\gamma) \leq c \ell_d(\gamma)$ for any path $\gamma \subset X$, and that $\delta_\Omega(x) = \delta'_\Omega(x) \leq c \delta_\Omega(x)$ for all $x \in \Omega$. Let $x, y \in \Omega$, $\gamma$ a geodesic in $(\Omega, k_\Omega)$ connecting $x$ to $y$, and $\gamma'$ a geodesic in $(\Omega, k'_\Omega)$ joining $x$ and $y$. Then

$$k'_\Omega(x, y) \leq \int_\gamma \frac{1}{\delta'_\Omega(z)} |dz| \leq \int_\gamma \frac{1}{\delta_\Omega(z)} c |dz| = c k_\Omega(x, y),$$

and

$$k_\Omega(x, y) \leq \int_\gamma \frac{1}{c \delta_\Omega(z)} |dz| \leq \int_\gamma \frac{c}{\delta'_\Omega(z)} |dz| = c k'(x, y).$$

(iii): For any $x \in \Omega$, we prove the claim. First assume $a$-quasim"obius with $\eta_1$-quasim"obius homeomorphism $f : (\partial'\Omega, d_x,\epsilon) \to (\partial'_\Omega, d'_x,\epsilon')$ with $\eta_1 = \eta_1(\delta, c, \epsilon, \epsilon')$. With this claim, the identity map $g : (\partial'\Omega, d) \to (\partial'\Omega, d')$ is used to construct the desired natural map $\phi' := g \circ \phi \circ f^{-1}$. Now we prove the claim. First assume $\epsilon' \geq \epsilon$. Let $f_0 : (\Omega, k_\Omega) \to (\Omega, k'_\Omega)$ be the identity map. Since by (iii) $f_0$ is $c$-bilipschitz, the associated boundary map $\partial f_0 : (\partial'\Omega, d_x,\epsilon) \to (\partial'_\Omega, d'_x,\epsilon')$ is $\eta_2$-quasim"obius with $\eta_2 = \eta_2(\delta, c, \epsilon)$. By Lemma 5.5 the identity map $p : (\partial'_\Omega, d'_x,\epsilon') \to (\partial'_\Omega, d'_x,\epsilon')$ is $\eta_3$-quasim"obius with $\eta_3 = \eta_3(\delta, \epsilon')$. It follows that $f = p \circ \partial f_0$ is $\eta_1 = \eta_3 \circ \eta_2$-quasim"obius with $\eta_1 = \eta_1(\delta, c, \epsilon, \epsilon')$. The claim is similarly proved when $\epsilon' < \epsilon$: $f$ in this case is the composition of the identity map $(\partial'\Omega, d_x,\epsilon) \to (\partial'\Omega, d'_x,\epsilon')$ and the boundary map $(\partial'\Omega, d_x,\epsilon) \to (\partial'_\Omega, d'_x,\epsilon')$ of the identity map $(\Omega, k) \to (\Omega, k')$. 

Theorem 9.1 follows from Lemma 9.2, and Theorem 9.3 Theorem 9.5 below.

**Theorem 9.3.** Let $(X, d)$ be a $C_a$-annular convex proper geodesic metric space, and $\Omega \subset X$ an unbounded rectifiably connected open subset with $\partial \Omega \neq \emptyset$. Suppose $(\Omega, k_\Omega)$ is $\delta$-hyperbolic and that there is an $\eta$-quasim"obius natural map $\phi : (\partial'\Omega, d_w,\epsilon) \to (\partial'\Omega, d)$. Then $(\Omega, d)$ is $c_1$-uniform with $c_1 = c_1(C_a, \delta, \eta, \epsilon)$.

**Proof.** Let $x_1, x_2 \in \Omega$, and $\gamma : x_1 \leadsto x_2$ be a quasihyperbolic geodesic. By Theorem 7.4, Lemma 7.6, and the existence of a natural map there is a quasihyperbolic geodesic line $\alpha : a_1 \leadsto a_2$ with $a_1, a_2 \in \partial'\Omega$ such that for $i = 1, 2$, $k_\Omega(x_i, |\alpha|) \leq C = C(\delta, C_a)$. Similarly exist points $w_1, w_2 \in |\alpha|$ satisfying $k_\Omega(x_i, w_i) \leq C$. Let $f : |\gamma| \to |\alpha|$ be a length map with $f(x_1) = w_1$. Then by Lemma 3.5, for every $x \in |\gamma|$ we have $k_\Omega(f(x), x) \leq C$. We will show that $\gamma$ is a uniform curve. By Lemma 2.5, we may assume that

$$d(x_1, x_2) \geq \max\{\delta_\Omega(x_1), \delta_\Omega(x_2)\}. \quad (21)$$
We first demonstrate that $\ell_d(\gamma_x) \wedge \ell_d(\gamma_{xx}) \leq C \delta_\Omega(x)$ for all $x \in \gamma$. If $a_2 = \infty$, then by Lemma 8.3, $\ell_d(\alpha) \leq \ell_d(\alpha_{a_2}) \leq C \delta_\Omega(f(x))$. Hence by Lemma 2.7, as $k_\Omega(f(z), z) \leq C$ for all $z \in [\gamma]$, we have

$$\ell_d(\gamma_x) \leq C \ell_d(\alpha(\gamma_x)) \leq C \delta_\Omega(f(x)) \leq C e^C \delta_\Omega(f) = C \delta_\Omega(x).$$

We obtain a similar inequality if $a_1 = \infty$. Now we assume that $a_1 \neq \infty \neq a_2$, and let $\xi_\alpha \in [\alpha]$ be the point given by Lemma 8.5. After Switching $a_1$ and $a_2$ if necessary, we may assume $f(x) \in \alpha_\alpha(\xi_\alpha)$. We have $f(x) \in \alpha_{a_1}(f(x))$ for some $i \in \{1, 2\}$. By Lemma 8.5 (ii),

$$\ell_d(\alpha(\gamma_x)) \leq C \delta_\Omega(f(x)) \leq C e^C \delta_\Omega(x).$$

Again by Lemma 2.7, we have

$$\ell_d(\gamma_x) \leq C \delta_\Omega(x).$$

This completes the proof that $\gamma$ satisfies the second condition for a uniform curve.

Finally, we need to prove that $\ell_d(\gamma) \leq C d(x_1, x_2)$. We break the proof up into two cases.

**Case 1:** In this case we consider the possibility that

$$2d(f(x_1), f(x_2)) \geq \max\{\delta_\Omega(f(x_1)), \delta_\Omega(f(x_2))\}.$$

Note that by Lemma 2.2, as $k_\Omega(f(x_1), x_i) \leq C$, we have $d(f(x_i), x_i) \leq e^C \delta_\Omega(x_i)$. If $a_1 = \infty$ or if $a_2 = \infty$, then Lemma 2.7 together with Lemma 8.3 now implies that

$$\frac{1}{C} \ell_d(\gamma) \leq \ell_d(\alpha(\gamma_x)) \leq C \max\{\delta_\Omega(f(x_1)), \delta_\Omega(f(x_2))\} \leq 2C d(f(x_1), f(x_2))$$

$$\leq 2C [d(f(x_1), x_1) + d(x_1, x_2) + d(f(x_2), x_2)]$$

$$\leq 2C [e^C \delta_\Omega(x_1) + e^C \delta_\Omega(x_2) + d(x_1, x_2)].$$

By the basic assumption of (21) we made at the beginning of the proof, we now get

$$\ell_d(\gamma) \leq 2C^2 [2e^C d(x_1, x_2) + d(x_1, x_2)] = C d(x_1, x_2),$$

and we are done. If $a_1 \neq \infty \neq a_2$, then Lemma 2.7 together with Lemma 8.5 (ii) shows that

$$\frac{1}{C} \ell_d(\gamma) \leq \ell_d(\alpha(\gamma_x)) \leq C d(f(x_1), f(x_2)).$$

Now using the fact again that $d(f(x_i), x_i) \leq e^C \delta_\Omega(x_i)$ and inequality (21), we obtain the desired inequality

$$\ell_d(\gamma) \leq C d(x_1, x_2),$$

completing the proof in Case 1.

**Case 2:** Here we consider the case

$$2d(f(x_1), f(x_2)) < \max\{\delta_\Omega(f(x_1)), \delta_\Omega(f(x_2))\} = \delta_\Omega(f(x_2)).$$

Then $f(x_1) \in \overline{B}(f(x_2), \delta_\Omega(f(x_2))/2)$. Hence a geodesic $\beta$ with respect to the metric $d$ joining $f(x_1)$ and $f(x_2)$ is a 1-uniform curve (see Lemma 2.5), and

$$k_\Omega(f(x_1), f(x_2)) \leq \int_\beta \frac{1}{\delta_\Omega(x)} |dx| \leq \frac{2}{\delta_\Omega(f(x_2))} \ell_d(\beta) = \frac{2}{\delta_\Omega(f(x_2))} d(f(x_1), f(x_2)) < 1.$$
Therefore, \( k_\Omega(x_1, x_2) = k_\Omega(f(x_1), f(x_2)) \leq 1 \). Hence \( \ell_{k_\Omega}(\gamma) \leq 1 \), and by Lemma 2.2,

\[
\ell_d(\gamma) \leq \epsilon \delta_\Omega(x_1) \leq C d(x_1, x_2),
\]

where we again used the assumption (21) at the end. Thus the desired estimate is proved, concluding the proof that \( \gamma \) is a uniform curve. As the constants found above depend solely on \( \delta, C, \eta, \epsilon \), we have completed the proof. \( \square \)

**Lemma 9.4.** Let \((X, d)\) be a \( c\)-quasiconvex \( c\)-annular convex metric space, and \( \Omega \subset X \) be an open subset. If \( \partial \Omega = \{p\} \), then \( (\Omega, d) \) is \( 6c^2 \)-uniform.

**Proof.** Let \( x, y \in \Omega \) and \( t = d(x, p) \). Without loss of generality, \( d(x, p) \leq d(y, p) \).

First assume that \( d(y, p) \leq 2t \). The annular convexity of \((X, d)\) implies that there is a path \( |\gamma| \subset B(p, 2ct) \setminus B(p, 2t/c) \) such that its length \( \ell(\gamma) \leq c \delta(x, y) \leq 3ct \). Observe that \( |\gamma| \subset \Omega \). For any \( z \in \gamma \) we have

\[
\delta_\Omega(z) = d(z, p) \geq \frac{t}{c} \geq \frac{\ell(\gamma)}{3c^2},
\]

and hence \( \gamma \) is a \( 3c^2 \)-uniform curve.

Now assume \( d(y, t) > 2t \). Then \( d(y, p)/2 \leq d(x, y) \leq 2d(y, p) \). Let \( n \geq 2 \) be the integer with \( 2^{n-1}t < d(y, p) \leq 2^n t \). Choose a path \( \gamma \) joining \( x \) to \( y \) parametrized from \( x \) to \( y \), and for \( 1 \leq i \leq n-1 \) let \( x_i \) be the first point on \( \gamma \) with \( d(x_i, p) = 2^it \). Set \( x_0 = x \) and \( x_n = y \). Then \( x_i \in \Omega \). Let \( \gamma_i \) be a path with the properties that \( |\gamma_i| \subset B(p, c2^it) \setminus B(p, 2^it/c) \), has endpoints \( x_{i-1}, x_i \), and \( \ell(\gamma_i) \leq c \delta(x_{i-1}, x_i) \leq 3c2^{i-1}t \). Such a path is guaranteed by the quasiconvexity property together with the annular convexity of \( X \). Let \( \gamma' \) be the concatenation of the \( \gamma_i \). Then

\[
\ell(\gamma') = \sum_{i=1}^n \ell(\gamma_i) \leq \sum_{i=1}^n 3c2^{i-1}t \leq 3ct 2^n \leq 6c d(y, p) \leq 12c d(x, y).
\]

Similarly, \( \sum_{i=1}^k \ell(\gamma_i) \leq 3ct 2^k \). Let \( z \in \gamma_k \). Then

\[
\delta_\Omega(z) = d(z, p) \geq \frac{2^kt}{c} \geq \frac{1}{3c^2} \sum_{i=1}^k \ell(\gamma_i) \geq \frac{1}{3c^2} \ell(\gamma'_z).
\]

**Theorem 9.5.** Let \((X, d)\) be a \( c\)-annular convex proper geodesic space, and \( \Omega \subset X \) a bounded rectifiably connected open subset with \( \partial \Omega \neq \emptyset \). Suppose that \((\Omega, k_\Omega)\) is \( \delta \)-hyperbolic, and that the natural map \( \phi : (\partial^* \Omega, d_{w, \epsilon}) \to (\partial \Omega, d) \) exists and is \( \eta \)-quasimöbius. Then \((\Omega, d)\) is \( c_1 \)-uniform with \( c_1 = c_1(c, \delta, \eta, \epsilon) \).

**Proof.** By Lemma 9.4, if \( \partial \Omega \) consists of a single point, then \( \Omega \) is \( 6c^2 \)-uniform. It therefore suffices to assume that \( \partial \Omega \) contains at least two points. Fix some \( p \in \partial \Omega \) and consider \((I_p(X), d_p)\). By Theorem 4.1, \((I_p(X), d_p)\) is both \( c'\)-quasiconvex and \( c'\)-annular convex with \( c' = c'(c) \). Since \((\Omega, k_\Omega)\) is \( \delta \)-hyperbolic, Proposition 4.2 implies that \((\Omega, k_{\Omega,p})\) is \( \delta' \)-hyperbolic with \( \delta' = \delta'(\delta, c) \). Let \( \epsilon' = \epsilon'(\delta, c) := \min\{\epsilon_0(\delta), \epsilon_0(\delta')\} \). Proposition 5.6 implies that the boundary map \( \partial f : (\partial^* \Omega, d'_{w, \epsilon'}) \to (\partial^* \Omega, d_{w, \epsilon}) \) of the identity map \( f : (\Omega, k_{\Omega,p}) \to (\Omega, k_{\Omega}) \) is \( \eta_1 \)-quasimöbius with \( \eta_1 = \eta_1(\delta, c) \). By Lemma 5.5 the natural identity map, denoted in this proof by \( \text{id} : (\partial^* \Omega, d_{w, \epsilon}) \to (\partial^* \Omega, d_{w, \epsilon}) \), is \( \eta_2 \)-quasimöbius with \( \eta_2(t) = 4^{1+ \frac{1}{2}} t^{1/4} \).
Then the map \( \phi' = \phi \circ id \circ \partial f \) is a natural map for \( (\Omega, d_p) \) and is \( \eta' \)-quasimöbius with \( \eta' = \eta'(\delta, c, \epsilon, \eta) := \eta \circ \eta_2 \circ \eta_1 \). Since \( (\Omega, d_p) \) is unbounded, Theorem 9.3 now implies that \( (\Omega, d_p) \) is \( c'' \)-uniform with \( c'' = c''(\delta', c', \epsilon', \eta') = c''(\delta, c, \epsilon, \eta) \). Now the theorem follows from Theorem 4.1 (3).

\( \square \)

10 An application to quasimöbius maps

In this section we show that quasimöbius maps between domains in annular convex metric spaces preserves uniformity. This result is quantitative.

**Theorem 10.1.** For \( i = 1, 2 \) let \( (X_i, d_i) \) be a proper metric space and \( \Omega_i \subset X_i \) an open subset with \( \partial \Omega_i \neq \emptyset \). Let \( h : \Omega_1 \to \Omega_2 \) be an \( \eta \)-quasimöbius homeomorphism. If \( \Omega_1 \) is \( c_1 \)-uniform and \( (X_2, d_2) \) is \( c_2 \)-quasiconvex and \( c_2 \)-annular convex, then \( \Omega_2 \) is \( c \)-uniform with \( c = c(c_1, c_2, \eta) \).

We first look at an example. Let \( \Delta \subset \mathbb{R}^2 \) be the open unit disk in the plane with the Euclidean metric \( d \). Then \( \Delta \) is 2-uniform. For any \( 0 < \epsilon < 1 \), the identity map \( (\Delta, d) \to (\Delta, d') \) is \( \eta \)-quasimöbius with \( \eta(t) = t' \), but \( (\Delta, d') \) is not uniform since there are no non-constant rectifiable curves in \( (\Delta, d') \). This example shows that it is reasonable to assume quasiconvexity of the target space, or at least one needs that the domains are rectifiably connected.

Now we begin the proof of Theorem 10.1. Let \( i \in \{1, 2\} \). If \( \Omega_i \) is bounded, set \( X'_i = X_i \) and \( d'_i = d_i \); if \( \Omega_i \) is unbounded, then fix any base point \( p_i \in \partial \Omega_i \) and set \( X'_i = S_{p_i}(X_i) \) and \( d'_i = d_{i,p_i} \). Denote by \( \partial \Omega'_i \) the boundary of \( \Omega_i \) in \( (X'_i, d'_i) \) and \( \overline{\Omega_i} \) the closure of \( \Omega_i \) in \( (X'_i, d'_i) \).

Let \( f_i : (\Omega_i, d_i) \to (\Omega_i, d'_i) \) be the identity map, and set \( h' := f_2 \circ h \circ f_1^{-1} : (\Omega_1, d'_1) \to (\Omega_2, d'_2) \). Let \( \eta_0(t) = 16t \).

**Lemma 10.2.** The map \( h' \) extends to a \( \eta' \)-quasimöbius homeomorphism \( \overline{\Omega_1} \to \overline{\Omega_2} \), which is still denoted by \( h' \). Here \( \eta' = \eta_0 \circ \eta \circ \eta_0 \). In particular, there exist \( a_1 \in \partial \Omega_1, a_2 \in \partial \Omega_2 \) such that for any \( \{x_i\} \subset \Omega_i \) with \( x_i \to a_1 \) we have \( h'(x_i) \to a_2 \).

**Proof.** The fact that \( f_i \) is \( \eta_0 \)-quasimöbius implies that the map \( h' : (\Omega_1, d'_1) \to (\Omega_2, d'_2) \) is \( \eta' \)-quasimöbius. Since \( \text{diam}(\Omega_i, d'_i) < \infty \), the map \( h' \) is a quasisymmetric map. Notice that \( (X'_i, d'_i) \) is proper. By Theorem 6.12 of [V2], \( h' \) extends to a quasisymmetric map between the closures of the domains. The continuity implies that the extension is also \( \eta' \)-quasimöbius. Now the lemma follows. \( \square \)

If \( \partial \Omega'_1 \) is a single point, then \( \partial \Omega'_2 \) and \( \partial \Omega_2 \) are also single points. By Lemma 9.4 \( (\Omega_2, d_2) \) is \( 6c^2 \)-uniform. From now on, we assume that \( \partial \Omega'_1 \) has at least two points.

Now we fix \( a_1 \in \partial \Omega'_1, a_2 \in \partial \Omega'_2 \) with the property stated in Lemma 10.2. Let \( X''_i = I_{a_i}(X'_i) \) and \( d''_i = (d'_i)_{a_i} \). Denote by \( \partial \Omega''_i \) the boundary of \( \Omega_i \) in \( (X''_i, d''_i) \) and \( \overline{\Omega''_i} \) the closure of \( \Omega_i \) in \( (X''_i, d''_i) \). It is seen that \( \partial \Omega''_i = \partial \Omega'_i \setminus \{a_i\} \) and \( \overline{\Omega''_i} = \overline{\Omega'_i} \setminus \{a_i\} \) as sets. Let

\[
g_i : (X'_i \setminus \{a_i\}, d'_i) \to (X''_i \setminus \{a_i\}, d''_i)\]
be the identity map and set

\[ h'' := g_2 \circ h' \circ g_1^{-1} : (\Omega'_1, d''_1) \to (\Omega''_2, d''_2). \]

Since \( g_i \) is \( \eta_0 \)-quasimöbius, \( h'' \) is \( \eta'' \)-quasimöbius, where \( \eta'' := \eta_0 \circ \eta' \circ \eta_0 \). The choice of \( a_1 \) and \( a_2 \) implies that for any \( \{x_i\} \subset \Omega_1 \) with \( d''_1(x_i, x_1) \to \infty \) we have \( d''_2(h''(x_i), h''(x_1)) \to \infty \). It follows that \( h'' \) is a \( \eta'' \)-quasimöbius homeomorphism.

Since \( (\Omega_1, d'_1) \) is \( c_1 \)-uniform, Theorem 4.6 (2) implies that \( (\Omega_1, d'_i) \) is \( c'_1 \)-uniform with \( c'_i = c'_1(c_1) \). Since \( \partial \Omega'_1 \) contains at least two points and \( a_1 \in \partial \Omega'_1 \), it follows from Theorem 4.1 (4) that \( (\Omega_1, d''_i) \) is \( c''_i \)-uniform with \( c''_i = c''(c'_1) = c''(c_1) \). Let \( k_i \) be the quasihyperbolic metric on \((\Omega_i, d''_i)\). By Theorem 6.1, \((\Omega_1, k_1)\) is \( \delta_1 \)-hyperbolic with \( \delta_1 = \delta_1(c''_1) = \delta_1(c_1) \). Let \( \epsilon_1 = \epsilon_1(c''_1) \) be as in Theorem 6.2. Then for any \( \epsilon \) satisfying \( 0 < \epsilon \leq \epsilon_1 \), Theorem 6.2 implies that there is a natural map \( \phi_1 : (\partial k_1, \Omega_1, k_{1, w_1, \epsilon}) \to (\partial \Omega'_1, d'_1) \) and the natural map is \( \eta_1 \)-quasimöbius with \( \eta_1 = \eta_1(c'_1, \epsilon) = \eta_1(c_1, \epsilon) \).

The fact that \( (\Omega_1, d''_i) \) is \( c''_i \)-uniform implies that \((\Omega''_1, d''_i)\) is \( c''_i \)-quasiconvex. On the other hand, since \( (X_2, d'_2) \) is \( c_2 \)-quasiconvex and \( c_2 \)-annular convex, Theorem 4.6 (3) and Theorem 4.1 (1) imply that \((X'_2, d'_2)\) is both \( c''_2 \)-quasiconvex and \( c''_2 \)-annular convex with \( c''_2 = c''_2(c_2) \).

**Lemma 10.3 (Lemma 2.3 of [V2]).** Suppose that \( X \) is \( \lambda_1 \)-quasiconvex, that \( q > 0 \), \( \lambda_2 \geq 0 \), and that \( f : X \to Y \) is a map such that \( d(f(x), f(y)) \leq \lambda_2 \) whenever \( d(x, y) \leq q \). Then \( d(f(x), f(y)) \leq (\lambda_1 \lambda_2/q)d(x, y) + \lambda_2 \) for all \( x, y \in X \).

**Lemma 10.4.** For \( i = 1, 2 \) let \( (Y_i, d_i) \) be a proper metric space and \( \Omega_i \subset Y_i \) a rectifiably connected open subset with \( \partial \Omega_i \neq \emptyset \). Suppose that \( Y_i \) is \( c''_i \)-quasiconvex and that there is an \( \eta'' \)-quasimöbius homeomorphism \( g : (\Omega_1, d_1) \to (\Omega_2, d_2) \). Let \( k_i \) be the quasihyperbolic metric on \((\Omega_i, d_i)\). Then the map \( g : (\Omega_1, k_1) \to (\Omega_2, k_2) \) is an \( (L, A) \)-quasi-isometry with \( L \) and \( A \) depending only on \( c''_1 \), \( c''_2 \) and \( \eta'' \).

**Proof.** By symmetry we only need to show that there exist constants \( L \) and \( A \) depending only on \( \eta'' \) and \( c''_2 \) such that \( k_2(g(x), g(y)) \leq Lk_1(x, y) + A \) for all \( x, y \in \Omega_1 \). Since \((\Omega_1, k_1)\) is a geodesic space, by Lemma 10.3 it suffices to find a constant \( q \) depending only on \( \eta'' \) and \( c''_2 \) such that \( k_2(g(x), g(y)) \leq 1 \) whenever \( k_1(x, y) \leq q \). We choose \( q \) to be the number

\[ q = \log \left( 1 + ( \eta''(2c''_2)^{-1}) \right). \]

Notice that \( q \) depends only on \( \eta'' \) and \( c''_2 \). We next show \( q \) has the required property.

Since \( Y_1 \) and \( Y_2 \) are proper and \( g : (\Omega_1, d_1) \to (\Omega_2, d_2) \) is an \( \eta'' \)-quasimöbius map, Theorem 6.12 of [V2] implies that \( g \) extends to a \( \eta'' \)-quasimöbius map \((\Omega_1, d_1) \to (\Omega_2, d_2)\), which is also denoted by \( g \). Let \( x, y \in \Omega_1 \) with \( k_1(x, y) \leq q \). Then

\[ q \geq k_1(x, y) \geq \log(1 + \frac{d_1(x, y)}{d_1(x) \wedge d_1(y)}) \geq \log(1 + \frac{d_1(x, y)}{\delta_1(x)}), \]

where \( \delta_1(z) = d_i(z, \partial \Omega_i) \) for any \( z \in \Omega_i \). It follows that \( d_1(x, y) \leq (e^q - 1) \delta_1(x) \). Let \( z \in \partial \Omega_1 \) with \( \delta_2(g(x)) = d_2(g(x), g(z)) \). Since \( g \) is \( \eta'' \)-quasimöbius, we have

\[ \frac{d_2(g(x), g(y))}{\delta_2(g(x))} \leq \frac{d_2(g(x), g(y))}{\delta_2(g(x), g(z))} \leq \frac{d_1(x, y)}{d_1(x, z)} \leq \frac{d_1(x, y)}{\delta_1(x)} \leq \eta''(e^q - 1) = \frac{1}{2c''_2}. \]
Since \((Y_2, d_2)\) is \(c''_p\)-quasiconvex, we can find a path \(\gamma\) connecting \(g(x)\) and \(g(y)\) such that \(\ell(\gamma) \leq c''_p d_2(g(x), g(y))\). It follows that \(\ell(\gamma) \leq c''_p d_2(g(x), g(y)) \leq \delta_2(g(x))/2\), and hence \(\delta_2(z) \geq \delta_2(g(x))/2\) for all \(z \in \gamma\). Therefore,

\[
k_2(g(x), g(y)) \leq \frac{1}{\delta_2(z)} |dz| \leq \frac{2}{\delta_2(g(x))} \ell(\gamma) \leq 1.
\]

\[QED\]

Lemma 10.4 implies that \(h'' : (\Omega_1, k_1) \to (\Omega_2, k_2)\) is a \((L, A)\)-quasi-isometry with \(L\) and \(A\) depending only on \(c''_p\), \(c_2\) and \(\eta''\). Since \((\Omega_1, k_1)\) is \(\delta_1\)-hyperbolic, \((\Omega_2, k_2)\) is \(\delta_2\)-hyperbolic with \(\delta_2 = \delta_2(\delta_1, L, A) = \delta_2(c_2, c_2, \eta) = \delta_2(c_1, c_2, \eta)\). Set \(\epsilon_2 = \epsilon_2(c_1, c_2, \eta) := \min\{\epsilon_1, \epsilon_0(\delta_2)\}\).

By Proposition 5.10, the boundary map \(\partial h'' : (\partial k_1 \Omega_1, k_1 w_{1,2}) \to (\partial k_2 \Omega_2, k_2 w_{2,2})\) of the map \(h'' : (\Omega_1, k_1) \to (\Omega_2, k_2)\) is \(\eta''\)-quasimöbius with \(\eta'' = \eta''(\delta_1, L, A) = \eta''(c_1, c_2, \eta)\). It follows that \(\phi_2 := h' \circ \phi_1 \circ (\partial h'')^{-1}\) is a natural map of \((\Omega_2, d_2)\) that is \(\eta_2\)-quasimöbius for some \(\eta_2 = \eta_2(c_1, c_2, \eta) := \eta' \circ \eta \circ \eta_3\), where \(\eta_3\) depends only on \(\eta''\). Now Theorem 9.3 implies that \((\Omega_2, d_2)\) is \(c'\)-uniform with \(c' = c'(c''_p, \delta_2, c_2, \eta_2) = c'(c_1, c_2, \eta)\). Now the result follows from Theorem 4.1 (3) and Theorem 4.6 (4).

The proof of Theorem 10.1 is now complete.

### 11 Two examples and one question

We give two examples that show the conclusion of the main theorem may fail if the space \(X\) is not quasiconvex and annular convex.

**Example 1.** The space \(X\) is a subset of \(\mathbb{R}^2\). Let \(B_1\) be the graph of \(y = x \sin \frac{1}{x}, -1 \leq x < 0\), \(B_2\) the graph of \(y = (x - 1) \sin \left(\frac{1}{x-1}\right), 1 < x \leq 2\), \(B_3 = \{(x, y) : x = -1, \sin(1) \leq y \leq 2\}\), \(B_4 = \{(x, y) : x = 2, \sin(1) \leq y \leq 2\}\) and \(B_5 = \{(x, y) : -1 \leq x \leq 2, y = 2\}\). Let \(\Omega = \bigcup_{i=1}^5 B_i\) and \(X = \Omega \cup \{(0, 0), (1, 0)\}\). We equip \(X\) with the Euclidean metric. We notice that \(X\) is homeomorphic to \([0, 1]\), \(\Omega\) is homeomorphic to \((0, 1)\) and \(\partial \Omega = \{(0, 0), (1, 0)\}\).

The space \((\Omega, k_\Omega)\) is isometric to the real line and hence is hyperbolic; \(\partial^*\Omega\) consists of two points. The natural map \((\partial^*\Omega, k_{\partial^*\Omega}) \to (\partial \Omega, d)\) exists and is trivially quasimöbius, but \((\Omega, d)\) is not uniform. Indeed, for \(x, y \in \Omega\), let \(\gamma_{xy}\) be the (unique) arc in \(\Omega\) connecting \(x\) to \(y\); then \(\ell(\gamma_{xy}) \to \infty\) as \(x \to (0, 0)\) and \(y \to (1, 0)\) while \(d(x, y) \leq 2\). The metric space \((X, d)\) is not quasiconvex.

**Example 2.** Let \(n \geq 1\) be an integer and set

\[
X = (\{-n, n\} \times \{0\}) \cup (\{0\} \times [-1, n]) \subset \mathbb{R}^2.
\]

Let \(p_1 = (n, 0), p_2 = (-n, 0), p_3 = (0, n)\) and \(p_4 = (0, -1)\). With \(X\) be equipped with the path metric, we choose \(\Omega = X \setminus \{p_1, p_2, p_3, p_4\}\). Then \(\partial \Omega = \{p_1, p_2, p_3, p_4\}\). The space \((\Omega, k_{\Omega})\) is a tree with 4 rays glued at the common vertex, hence it is 0-hyperbolic. Let \(w = (0, 0)\) be the origin. The natural map \(\phi : (\partial^*\Omega, d_{w,1}) \to (\partial \Omega, d)\) exists and is a bijection. Notice that for any quadruple \(Q\) of distinct points in \(\partial^*\Omega\), we have \(cr(Q, d_{w,1}) = 1\) and \(cr(Q, d) = 1\). It follows that \(\phi\) is \(\eta\)-quasimöbius with \(\eta(t) = t\). The domain \((\Omega, d)\) is \(n\)-uniform, but is not \(c\)-uniform for any \(c < n\): any path from \(p_1\) to \(p_2\) has to pass through \(w\), which is at distance 1
from \( p_4 \). Therefore the quantitative statement fails for \( \Omega \subset (X, d) \). Observe that the metric space \( (X, d) \) is geodesic but is not annular convex.

Given the main theorem of the paper and the above two examples, it is natural to ask the following question:

**Question.** Let \( (X, d) \) be a quasiconvex proper metric space and \( \Omega \subset X \) a rectifiably connected open subset with \( \partial \Omega \neq \emptyset \). Suppose \( (\Omega, k_\Omega) \) is Gromov hyperbolic, and the natural map exists and is quasimöbius. Is \( (\Omega, d) \) uniform?

Example 2 shows that one can not expect to control the uniformity constant even if the answer to the above question is yes.

**References**


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