Chapter 2

The Pendulum Equation

2.1 Derivation of the Pendulum Equation

Prerequisites:
- Basic calculus skills

Advanced Prerequisites:
- Classical mechanics

Learning Objectives:
- Know the equation of motion for a pendulum.
- Understand the assumptions on which the pendulum equation is based.
- Understand the role of kinematics, universal laws, and constitutive laws in mathematical modeling.

In this section we derive the equation of motion of a pendulum on a moving support. While our derivation can be applied to more general situations, we are particularly interested in the mechanical device shown in Figure 2.1. In this mechanism, the pendulum pivots on a support that can be moved in a periodic motion whose amplitude and frequency can be adjusted.

There is a three-step process for describing equations of physics that is often helpful in clarifying the distinction between different types of ideas. The first step is to describe the kinematics of the process, i.e. the basic variables in the problem and the physically inherent restrictions on them. In the case of the pendulum we simply describe all possible motions of an idealized pendulum. Next, one poses universal laws that govern all processes of the type under consideration. In our case, we describe Newton’s second law: the balance of linear momentum for a rigid body. Finally, one postulates constitutive laws that
differentiate one physical situation from another. In our case, this amounts to specifying the types of forces that are applied to the pendulum.

### 2.1.1 Kinematics

Let’s begin by distinguishing between a real pendulum and the mathematical idealizations we consider here. We will assume the following.

**Hypothesis 2.1.** The pendulum is a rigid body, with total mass \( m \), whose motion is described by the motion in time (denoted by \( t \in [0, \infty) \)) of its pivot \( p(t) \) and its center of mass \( x(t) \).

A few observations are in order about this first hypothesis.

- Most real pendulums are not rigid. Chains and ropes holding heavy weights are quite flexible. They act like rigid objects only when they are under strong tension.

- In many problems, the motion of the pivot is prescribed, and in the simplest problems the pivot is fixed. Of course, in some problems (for instance multiple, connected pendulums) the position of the pivot can be unknown.

- Since the pendulum is assumed rigid, the pivot and the center of mass must remain a fixed distance apart. We call this distance the **length** of the pendulum \( l \).
2.1. Derivation of the Pendulum Equation

- When we say that the motion of the pendulum is “described” by the motion of the pivot and the center of mass, we are simply saying that we are not interested in the rotation of the rigid body about the axis connecting the pivot and the center of mass. In general, a rigid body could rotate about this axis, but we won’t measure that motion and are not interested in describing it.

Since we wish to model the mechanism in Figure 2.1, we make the following assumption.

Hypothesis 2.2. We assume that the pivot $p$ and the center of mass $x$ are described by vectors in a plane spanned by a fixed orthonormal pair of vectors $e_1$ and $e_2$.

This will allow us to model problems that are confined to the plane by physical constraint (as Figure 2.1) or whose three-dimensional solution can be shown to reduce to two dimensions. While there are many interesting pendulum problems that take place in three dimensions, we will be able to observe enough complexity in two-dimensional problems to make the tradeoff of simplicity for generality worthwhile.

We think of the plane as depicted in Figure 2.2, with $e_1$ oriented down and $e_2$ a counter-clockwise quarter-turn away. We define

$$
p(t) = p_1(t)e_1 + p_2(t)e_2$$
$$
x(t) = x(t)e_1 + y(t)e_2$$

and

$$
x(t) - p(t) = lk_1(\theta(t))$$

Figure 2.2: Planar motion of a pendulum.
where
\[
\begin{align*}
  k_1(\theta) &= \cos \theta e_1 + \sin \theta e_2 \\
  k_2(\theta) &= -\sin \theta e_1 + \cos \theta e_2
\end{align*}
\]

We use the fact that the distance between the pivot and center of mass are fixed to compute the velocity and acceleration of the center of mass.

\[
\begin{align*}
  x'(t) &= p'(t) + lk_2(\theta(t))\theta'(t) \\
  x''(t) &= p''(t) + lk_2(\theta(t))\theta''(t) - lk_1(\theta(t))\theta'(t)^2
\end{align*}
\]

The linear momentum of the pendulum is defined to be \(mx'(t)\).

### 2.1.2 Universal law: Balance of linear momentum

Here we state a version of Newton’s second law that applies to rigid bodies. We assume that there is a vector \(f(t) = f_1(t)e_1 + f_2(t)e_2\) that describes the net force applied to the rigid body. Then the balance of linear momentum is as follows.

**Hypothesis 2.3.** The time rate of change of linear momentum is equal to the net force applied to the pendulum, i.e.
\[
mx'' = f
\]

If we write
\[
\begin{align*}
  p''(t) &= \alpha_n(t)k_1(\theta(t)) + \alpha_t(t)k_2(\theta(t)) \\
  f(t) &= f_n(t)k_1(\theta(t)) + f_t(t)k_2(\theta(t))
\end{align*}
\]

then we can write the balance law as two scalar equations.

\[
\begin{align*}
  -ml(\theta')^2 + m\alpha_n &= f_n \\
  ml\theta'' &= f_t - m\alpha_t
\end{align*}
\]

At this point, we have two scalar equations in three scalar unknowns: \(\theta\), \(f_n\), and \(f_t\). (The quantities \(m\), \(l\), \(\alpha_n\) and \(\alpha_t\) are assumed to be prescribed.)

### 2.1.3 Constitutive laws: Applied forces

In this section, we describe the external forces on the pendulum. We will assume three types of forces:

1. The force of gravity \(f_g\),
2. Viscous or frictional forces \(f_v\),
3. The force of constraint applied by the pivot \(f_c\).
Hypothesis 2.4. Gravity is the most familiar of these forces. We assume that it acts in the positive $e_1$ direction, and that there is a gravitational constant $g$ such that the force of gravity on the pendulum is given by

$$f_g = mg e_1 = mg(\cos \theta k_1(\theta) - \sin \theta k_2(\theta)) \quad (2.4)$$

Hypothesis 2.5. The second type of force is viscous or frictional. We assume that this force is tangential to the motion of the pendulum and takes the form

$$f_v = f_v(\theta, \theta') k_2(\theta) \quad (2.5)$$

where $f_v$ is a smooth function. The function $f_v$ depends on the angle $\theta$ (the pivot might be particularly “sticky” at particular angles) and the angular velocity $\theta'$. We make the following general assumptions about the behavior of $f_v$:

$$f_v(\theta, 0) = 0 \quad (2.6)$$

$$\frac{\partial f_v}{\partial \theta'}(\theta, \theta') < 0 \quad (2.7)$$

The first assumption (2.6) says that if the pendulum is not moving, there is no frictional force. The second says that $f_v$ is a monotone decreasing function of the angular velocity.

Problem 2.1. Show that (2.6) and (2.7) imply that the frictional force acts in the opposite direction to the velocity of the pendulum. In particular, show that

$$\omega f_v(\theta, \omega) \leq 0.$$

Draw a generic graph of $\omega \mapsto f_v(\theta, \omega)$ for fixed $\theta$.

Remark 2.6. More general models of friction with discontinuous functions $f_v$ are probably more realistic (see, e.g. [? and reference therein]), but we will restrict ourselves to the smooth case here. The most common choice of a frictional force is a linear viscosity of the form

$$\tilde{f}_v = -\nu \theta' \quad (2.8)$$

where $\nu > 0$ is a constant.

Hypothesis 2.7. The final force is the one that the pivot exerts on the pendulum to keep it from flying away. To specify this, let’s first think of the pivot as fixed. Since the pendulum is rigid, it’s center of mass must move in a circle about the pivot. If the pendulum is to remain rigid, any force that tries to move the center of mass off the circle (i.e. in a direction normal to it) must be countered by an equal and opposite force exerted by the pivot. That is,

$$f_c = f_c k_1(\theta) \quad (2.9)$$

We assume that this holds even when the pivot moves. Under this assumption the normal component of the balance of forces (2.2) can be written in the form

$$f_c = \alpha_n - \text{im}(\theta')^2 - mg \cos \theta.$$

We assume that the forces of constraint $f_c$ respond so that this equation is satisfied identically, so we will not have to consider it any further in our analysis.
Definition 2.8. The remaining balance law (2.3) is now a single, scalar, second-order ordinary differential equation for $\theta$ called the pendulum equation.

$$ml\ddot{\theta} = -mg \sin \theta + f_v(\theta, \theta') - m\alpha_t$$  \hspace{1cm} (2.10)

Remark 2.9. It is often to our advantage to convert an ordinary differential equation or system of equations to a first-order system. This is done by introducing new variables to represent all “lower-order” derivatives of the original variables and introducing a new equation that defines each new variable. In this case, since second-order derivatives of $\theta$ are the highest, we introduce the new variable $\omega$ satisfying $\omega = \theta'$. Then we write the single second-order equation (2.10) in one unknown as the following first-order system of two equations in two unknowns.

$$\theta' = \omega$$  \hspace{1cm} (2.11)
$$\omega' = -\frac{g}{l} \sin \theta + \frac{1}{ml} (f_v(\theta, \omega) - m\alpha_t)$$  \hspace{1cm} (2.12)

Connections: Deeper information on the material in this section can be found in the following sections.

- Vibrating string
- Theory of ordinary differential equations

2.2 Abstract ODE Results

Prerequisites:

- Basic calculus skills
- Skills in language and logic, techniques of proof

Advanced Prerequisites:

- Familiarity with basic normed vector space techniques, contraction mapping principle

Learning Objectives:

- Understanding of the basic hypotheses needed for existence and uniqueness of solutions of ordinary differential equations.
- Familiarity with the cases in which the basic existence and uniqueness theorems fail.
- Understanding of the proof of Gronwall’s inequality and the uniqueness theorem for ordinary differential equations.
2.2. ABSTRACT ODE RESULTS

2.2.1 Existence

We begin with the basic existence result for initial value problems for ordinary differential equations.

Theorem 2.10 (ODE existence, Picard-Lindelöf). Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set, and let $F : D \rightarrow \mathbb{R}^n$ be continuous in its first variable and uniformly Lipschitz in its second; i.e., for $(t, y) \in D$, $F(t, y)$ is continuous as a function of $t$, and there exists a constant $\gamma$ such that for any $(t, y_1)$ and $(t, y_2)$ in $D$ we have

$$|F(t, y_1) - F(t, y_2)| \leq \gamma |y_1 - y_2|.$$  \hspace{1cm} (2.13)

Then, for any $(t_0, y_0) \in D$, there exists an interval $I := (t^-, t^+)$ containing $t_0$, and at least one solution $y \in C^1(I)$ of the initial-value problem

$$\frac{dy}{dt}(t) = F(t, y(t)), \hspace{1cm} (2.14)$$

$$y(t_0) = y_0. \hspace{1cm} (2.15)$$

Remark 2.11. The proof is given elsewhere in the series, but we make note of a few ideas of the proof here. One version of the proof uses an important technique for PDEs: the construction of an equivalent integral equation. In this proof, one shows that there is a continuous function $y$ that satisfies

$$y(t) = y_0 + \int_{t_0}^{t} F(s, y(s)) \, ds.$$  \hspace{1cm} (2.16)

Then the fundamental theorem of calculus implies that $y$ is differentiable and satisfies (2.14), (2.15) (cf. the results on smoothness below). The solution of (2.16) is obtained from an iterative procedure; i.e., we begin with an initial guess for the solution (usually the constant function $y_0$) and proceed to calculate

$$y_1(t) = y_0 + \int_{t_0}^{t} F(s, y_0) \, ds,$$
$$y_2(t) = y_0 + \int_{t_0}^{t} F(s, y_1(s)) \, ds,$$
$$\vdots$$
$$y_{k+1}(t) = y_0 + \int_{t_0}^{t} F(s, y_k(s)) \, ds,$$
$$\vdots$$  \hspace{1cm} (2.17)

Of course, to complete the proof one must show that this sequence converges to a solution. This is done using the Contraction Mapping Principle, which is discussed elsewhere in this series.
2.2.2 Uniqueness

In this section we derive an a priori estimate for solutions of ODEs that is related to the energy estimates for PDEs that we examine in later chapters. The uniqueness theorem (Theorem 2.14) is an immediate consequence of this result. To derive our estimate we need a fundamental inequality called Gronwall’s inequality.

Lemma 2.12 (Gronwall’s inequality). Let

\[ u : [a, b] \to [0, \infty), \]
\[ v : [a, b] \to \mathbb{R}, \]

be continuous functions and let \( C \) be a constant. Then if

\[ v(t) \leq C + \int_a^t v(s)u(s) \, ds \quad (2.18) \]

for \( t \in [a, b] \), it follows that

\[ v(t) \leq C \exp \left( \int_a^t u(s) \, ds \right) \quad (2.19) \]

for \( t \in [a, b] \).

Proof. We define \( W(t) := C + \int_a^t v(s)u(s) \, ds \). We now use the hypotheses that \( v \leq W \) and \( u \geq 0 \) to get

\[ W'(t) = v(t)u(t) \leq W(t)u(t). \]

Multiplying this inequality by \( \exp \left( -\int_a^t u(s) \, ds \right) \) and rearranging gives us

\[ \frac{d}{dt} \left[ W(t) \exp \left( -\int_a^t u(s) \, ds \right) \right] = (W'(t) - W(t)u(t)) \exp \left( -\int_a^t u(s) \, ds \right) \leq 0. \]

Integrating this from \( a \) to \( t \) gives us

\[ W(t) \exp \left( -\int_a^t u(s) \, ds \right) \leq W(0) = C \]

or

\[ v(t) \leq W(t) \leq C \exp \left( \int_a^t u(s) \, ds \right). \]

This completes the proof. \( \square \)

We now use Gronwall’s inequality to obtain an “energy estimate” for ordinary differential equations.
Lemma 2.13 (Energy estimate for ODEs). Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the hypotheses of Theorem 2.10, in particular let it be uniformly Lipschitz in its second variable with Lipschitz constant $\gamma$ (cf. (2.13)). Let $y_1$ and $y_2$ be solutions of (2.14) on the interval $[t_0, T]$; i.e.,

$$y'_i(t) = F(t, y_i(t))$$

for $i = 1, 2$ and $t \in [t_0, T]$. Then

$$|y_1(t) - y_2(t)|^2 \leq |y_1(t_0) - y_2(t_0)|^2 e^{2\gamma(t-t_0)}.$$ 

(2.20)

Proof. We begin by using the differential equation, the Cauchy-Schwarz inequality and the Lipschitz condition to derive the following inequality.

$$|y_1(t) - y_2(t)|^2 = |y_1(t_0) - y_2(t_0)|^2 + \int_{t_0}^{t} \frac{d}{ds} \left( y_1(s) - y_2(s) \right)^2 ds$$

$$\leq |y_1(t_0) - y_2(t_0)|^2 + \int_{t_0}^{t} 2|y_1(s) - y_2(s)| |F(s, y_1(s)) - F(s, y_2(s))| ds$$

$$\leq |y_1(t_0) - y_2(t_0)|^2 + \int_{t_0}^{t} 2\gamma |y_1(s) - y_2(s)|^2 ds.$$ 

Now (2.20) follows directly from Gronwall’s inequality. □

Note we if we set $y_1(t_0) = y_2(t_0)$ and use (2.20) we can immediately derive the following uniqueness result for ODEs.

Theorem 2.14 (ODE uniqueness). Let the function $F$ satisfy the hypotheses of Theorem 2.10. Then the initial-value problem (2.14), (2.15) has at most one solution.

Remark 2.15. Observe that this result is obtained \textit{a priori}: nothing we did depended on the existence of a solution, only on the equations that a solution would satisfy if it \textit{did} exist.

Example 2.16. It should also be noted that although this result covers a very wide range of initial-value problems, there are some standard, simple examples for which uniqueness fails. For instance, the problem

$$\frac{dy}{dt} = y^{1/3},$$

$$y(0) = 0$$
has an entire family of solutions parameterized by \( \gamma \in [0, 1] \):

\[
y_{\gamma}(t) := \begin{cases} 
0, & 0 \leq t \leq \gamma \\
\left(\frac{3}{2}(t - \gamma)\right)^{3/2}, & \gamma < t \leq 1.
\end{cases}
\]

The key here is that the function \( y \mapsto F(t, y) = y^{1/3} \) does not satisfy the Lipschitz condition (2.13) and \( y = 0 \). To see this we note that

\[
\frac{|F(t, y) - F(t, 0)|}{|y - 0|} = \frac{|y|^{1/3}}{|y|} = |y|^{-2/3} \to \infty \text{ as } y \to 0.
\]

Thus, there can be no constant \( \gamma \) such that \( |F(t, y) - F(t, 0)| \leq \gamma |y - 0| \).

The concept of Lipschitz continuity and its relationship to continuity and differentiability is explored in more detail elsewhere in this series.

**Example 2.17.** While we frequently expect the solution of a problem to be unique, there are many situations in which we expect multiple solutions. A common mathematical problem involving multiple solutions is an eigenvalue problem. You should, of course, be familiar with the various existence and multiplicity results from finite-dimensional linear algebra, but let’s consider the following second-order ODE depending on the parameter \( \lambda \):

\[
u'' + \lambda u = 0. \tag{2.21}
\]

Of course, if we imposed two initial conditions (at one point in space) Theorem 2.14 would imply that we would have a unique solution. (To apply the theorem directly we need to convert the problem from a second-order equation to a first-order system.) However, if we impose the two-point boundary conditions

\[
u(0) = 0, \tag{2.22}
\]
\[
u'(1) = 0, \tag{2.23}
\]

the uniqueness theorem does not apply. Instead we get the following result.

**Theorem 2.18.** There are two alternatives for the solutions of the boundary-value problem (2.21), (2.22), (2.23).

1. If \( \lambda = \lambda_n := ((2n + 1)^2 \pi^2)/4, n = 0, 1, 2, \ldots \), then the boundary-value problem has a family of solutions parameterized by \( A \in (-\infty, \infty) \):

\[
u_n(x) = A \sin \frac{(2n + 1)\pi}{2}x.
\]

In this case we say \( \lambda \) is an eigenvalue.

2. For all other values of \( \lambda \) the only solution of the boundary-value problem is the trivial solution

\[
u(x) \equiv 0.
\]
2.2.3 Stability

The term stability is one that has a variety of different meanings within mathematics. One often says that a problem is stable if it is “continuous with respect to the data”; i.e., a problem is stable if when we change the problem “slightly,” the solution changes only slightly. We make this precise below in the context of initial-value problems for ODEs.

We assume that $F$ satisfies the hypotheses of Theorem 2.10, and we define $\hat{y}(t,t_0,y_0)$ to be the unique solution of (2.14), (2.15). We then have the following standard result.

**Theorem 2.19 (Continuity with respect to initial conditions).** The function $\hat{y}$ is well defined on an open set

$$U \subset \mathbb{R} \times D.$$ 

Furthermore, at every $(t,t_0,y_0) \in U$ the function

$$\{(t_0,y_0) \mapsto \hat{y}(t,t_0,y_0)\}$$

is continuous; i.e., for any $\epsilon > 0$ there exists $\delta$ (depending on $(t,t_0,y_0)$ and $\epsilon$) such that if

$$|(t_0,y_0) - (\tilde{t}_0,\tilde{y}_0)| < \delta,$$

then $\hat{y}(t,\tilde{t}_0,\tilde{y}_0)$ is well defined and

$$|\hat{y}(t,t_0,y_0) - \hat{y}(t,\tilde{t}_0,\tilde{y}_0)| < \epsilon.$$ \hspace{1cm} (2.24)

Thus, we see that small changes in the initial conditions result in small changes in the solutions of the initial-value problem.

**Connections:** Deeper information on the material in this section can be found in the following sections.

- Contraction mapping theorem.
- Two-point boundary value problems.

2.3 Analysis of the Nonlinear Pendulum Equation

**Prerequisites:**

- Basic calculus skills.
- Skills in language and logic, techniques of proof.
- Familiarity with the derivation of the pendulum equation.
- Familiarity with the existence and uniqueness theorems for ODEs.
CHAPTER 2. THE PENDULUM EQUATION

Advanced Prerequisites:

- MATLAB, Mathematica, or Maple

Learning Objectives:

- Understanding of the proof of conservation of energy for the pendulum equation.
- The ability to construct phase portraits for simple ODEs.
- The ability to analyze the behavior of solutions of an ODE based on its phase portrait.

We now wish to analyze the pendulum equation

\[ ml\ddot{\theta} = -mg\sin \theta + f_v(\theta, \dot{\theta}) - \alpha_t \]

or (equivalently) the first order system

\[
\begin{align*}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\frac{g}{l}\sin \theta + \frac{1}{ml}(f_v(\theta, \omega) - m\alpha_t)
\end{align*}
\]

From the previous section, we know that the initial value problem has a unique solution that depends continuously on the initial conditions, but we would like more detailed information.

2.3.1 Conservation of energy

We begin with a conservation law which any solution of the pendulum equation must satisfy.

Theorem 2.20. Let \( \theta \in C^2(t_0, t_1) \) be any solution of the unforced pendulum equation \((\alpha_t \equiv 0)\). Then the energy

\[ E(t) := \frac{ml}{2}(\dot{\theta})^2 - mg\cos \theta \]

never increases, i.e.

\[ \frac{d}{dt} E(t) \leq 0. \]

If the equation has no damping (i.e. \( f_v \equiv 0 \)) then the energy \( E(t) \) is constant.

Proof. We simply use the product rule and the chain rule to directly compute

\[ \frac{d}{dt} E(t) = ml\ddot{\theta}\dot{\theta} + gl\sin \theta \dot{\theta} = \dot{\theta} f_v(\theta, \dot{\theta}) \leq 0 \]

Here we have used Problem 2.1 to get the inequality. If there is no damping we have \( \frac{d}{dt} E(t) \equiv 0. \) \( \Box \)
Problem 2.2. Consider the differential equation
\[ y'' + \alpha y' + \lambda y = 0 \]
Find a function \( E(y, y') \) such that if \( y(t) \) is a solution of the ODE above, then \( t \mapsto E(y(t), y'(t)) \) is nonincreasing if \( \alpha > 0 \), nondecreasing if \( \alpha < 0 \) and constant if \( \alpha = 0 \).

Problem 2.3. Let \( y : \mathbb{R} \to \mathbb{R} \) be a solution of
\[ y''' + y = 0 \]
What can you sat about the quantity \( \dot{E}(t) := y'''(t)y''(t) + y'(t)y(t) \)?

2.3.2 Phase-plane analysis

In this section, we use a technique called “phase-plane analysis” to gain some qualitative information about solutions of the pendulum equation without actually computing the solutions.

When we graph solutions of a scalar differential equation such as the pendulum equation in the “usual” way, we plot the solution on the vertical axis as a function of time, which runs on the horizontal axis. To do this for a general first-order system of \( n \) equations we would have to plot \( n \) separate curves. On the other hand, if \( n \) is two or three, we could represent the solution as a curve in \( n \)-space. A graph representing the family of curves produced by all solutions of initial-value problems is called the phase portrait of the differential equation.

Fortunately, it is sometimes easier to produce the phase portrait than to produce the actual solutions of the ODE. For instance, consider the case of the pendulum equation where there is no tangential force or viscosity and \( g = l \),
\[ \theta'' + \sin \theta = 0 \]
or, as a system,
\[ \begin{align*}
\theta' & = \omega \\
\omega' & = -\sin \theta
\end{align*} \]
According to our conservation of energy result, solutions \((\theta(t), \omega(t))\) of this system must satisfy
\[ \frac{\omega(t)^2}{2} - \cos \theta(t) = C \]
where \( C \) is a constant. Thus, if we graph trajectories of solutions in the \((\theta, \omega)\) phase-plane, the curves must lie on the level curves of the function
\[ f(\theta, \omega) := \frac{\omega^2}{2} - \cos \theta \]
Furthermore, since the unique solution of an initial value problem goes through each point in the \((\theta, \omega)\), every level curve represents the trajectory of a solution.
We can easily plot the level curves of $f$ with a computer graphics package such as MATLAB, Mathematica, or Maple as we do in Figure 2.3. Now that we have our phase plane portrait, how do we interpret it? First, consider how a solution would move along one of these curves as time increased. Recall that $\theta$ is the horizontal axis while $\omega = \theta'$ is the vertical axis. When $\omega = \theta' > 0$, i.e. above the $\theta$-axis, $\theta$ must increase in time, so we move to the right along a trajectory. When $\theta' < 0$, i.e. below the $\theta$-axis, $\theta$ must decrease in time, so we move to the left along a trajectory. With this in mind let us examine the several types of trajectories we see in the phase plane.

- There is a collection of closed orbits that would be traversed clockwise as time increased.
- There is a collection of orbits in the upper half-plane in which $\theta$ always increases.
- There is a collection of orbits in the lower half-plane in which $\theta$ always decreases.
- There is a curious collection of trajectories in which four separate curves seem to intersect at a single point.

We can give a physical interpretation of each of these trajectories.

The first set of trajectories are what we usually think of as "pendulum motion." The motion is periodic, with a maximum and minimum value of $\theta$ at
which the angular velocity is zero. The angular velocity is at its maximum (and minimum) as the pendulum swings by the origin \((\theta = 0)\).

The second and third sets of trajectories are a bit stranger. These occur when we give the pendulum enough of a “push” so that it swings over the top of the pivot. Our model (in which we assume no friction) predicts that the pendulum will swing round in a circle indefinitely, with minimum angular velocity as it goes over the top \((\theta = \pm \pi, \pm 3\pi, \pm 5\pi, \ldots)\) and maximum angular velocity as it passes the bottom \((\theta = 0, \pm 2\pi, \pm 4\pi, \ldots)\).

In order to examine the last set of curves, we first need to find the stationary points of the system: points \((\bar{\theta}, \bar{\omega})\) at which there is a time-independent solution to the differential equations \((\theta(t), \omega(t)) \equiv (\bar{\theta}, \bar{\omega})\). Since the time derivative must be zero we have

\[
\begin{align*}
\bar{\omega} &= 0 \\
-\sin \bar{\theta} &= 0
\end{align*}
\]

so that \(\bar{\theta} = 0, \pm \pi, \pm 2\pi, \pm 3\pi\). Thus the stationary points correspond to the pendulum in a vertical position, either at “rest” hanging straight down or precisely balanced standing straight up.

The points in the phase plane corresponding to the pendulum hanging straight down \((\omega = 0, \theta = 0, \pm 2\pi, \pm 4\pi, \ldots)\) are at the center of periodic orbits and are (appropriately) referred to as “centers.”

The points corresponding to the pendulum standing straight up \((\omega = 0, \theta = \pm \pi, \pm 3\pi, \ldots)\) lie at the intersection points of the trajectories referred to above. Note that the uniqueness theorem implies that no trajectory can go through a stationary point. (The only solution containing that point has to stay fixed for all time.) However, four trajectories lie on the same level curve of the energy as each of these critical points. Two of these curves (northeast and southwest) describe solutions that move away from the critical point as time increases. The other two (northwest and southeast) move toward the point. Note that each of these curves connect to another stationary point of the same type. (They lie on the same level set of the energy.) Physically, these curves describe a solution that leaves the perfectly balanced state, swings around once, and comes to rest (after infinite time) in the perfectly balanced state again.

Of course, this is not a likely scenario. And this notion is reflected in the fact that solutions with very similar initial conditions will have very different long term behavior. Start a solution with a slightly more negative initial velocity and the pendulum will spin clockwise forever. A slight increase in the initial \(\omega\) and it will spin counter clockwise. Leave \(\omega\) zero and slightly change the initial \(\theta\) and we get periodic solutions. (Note that this does not contradict our theorem on continuity with respect to initial data. All of these solutions are similar (pretty close to stationary) for a short time. They differ wildly in their long-term behavior.)

The other stationary points (the centers corresponding to the pendulum hanging straight down) have a very different behavior. They are surrounded by periodic orbits and small changes in initial conditions give solutions with
very similar orbits. In physical term we would say that the perfectly balanced pendulum as unstable while the pendulum hanging straight down is stable. We shall introduce a precise mathematical characterization of stability in another section.

Connections: Deeper information on the material in this section can be found in the following sections.

- Linear stability of systems of ODEs

### 2.4 Linearization of the Pendulum Equation

**Prerequisites:**

- Basic calculus skills
- Solution of second order linear ODEs with constant coefficients.

**Learning Objectives:**

- Understanding of the idea of linearizing a nonlinear ODE about a particular solution.
- Understanding of the concept of linear stability.

Let’s return to the general damped pendulum equation

\[ ml\ddot{\theta} = -mg\sin\theta + f_v(\theta, \theta') - m\alpha_t \]

and look for stationary points of the unforced equation. That is, we seek solutions where \( \theta = \theta_0 \) is constant in time or

\[ 0 = -mg\sin\theta_0 + f_v(\theta_0, 0). \]

Since, by hypothesis \( f_v(\theta_0, 0) = 0 \), the stationary points are just as they were for the undamped case \( \theta_0 = n\pi, n \in \mathbb{Z} \). As before, the points \( \theta_0 = 0, \pm 2\pi, \pm 4\pi, \ldots \) correspond to the pendulum hanging straight down while \( \theta_0 = \pm \pi, \pm 3\pi, \pm 5\pi, \ldots \) correspond to the pendulum balanced on its pivot.

We now want to look at solutions that are “close” to these equilibrium points. The technique we use is called “linearization.” This is a generalization of one of the most important tools in calculus: the use of the derivative to approximate a general curve by a tangent line. To see how linearization works, let \( f : \mathbb{R} \to \mathbb{R} \) be a given function and suppose we know that we know some particular solution of the equation

\[ f(x_0) = b_0 \]

and we would like to solve the “nearby” problem

\[ f(x) = b_0 + \epsilon \bar{b} \]
for $x$. Here $\bar{b} \in \mathbb{R}$ is given and $\epsilon$ is some small number. If $f$ is anything other than a very simple function, it will be impossible to get a closed form solution of the problem. However, we can use calculus to get an approximate solution. We look for a “nearby” solution of the form $x = x_0 + \epsilon \bar{x}$ (where $\epsilon$ is the small parameter appearing in the data and $\bar{x} \in \mathbb{R}$ is unknown). Approximating $f$ by its tangent line approximation we get

$$f(x) = f(x_0 + \epsilon \bar{x}) \approx f(x_0) + f'(x_0)\epsilon \bar{x}.$$  

We use this approximation in our equation and try to solve

$$f(x_0) + f'(x_0)\epsilon \bar{x} = b_0 + \epsilon \bar{b}.$$  

Note that the “zeroth” order term drop out since $f(x_0) = b_0$ while the terms with coefficient $\epsilon$ give us the equation

$$f'(x_0)\bar{x} = \bar{b}.$$  

(2.25)  

This is a linear equation for $\bar{x}$, and if $f'(x_0) \neq 0$ we easily get the solution

$$\bar{x} = \frac{\bar{b}}{f'(x_0)}.$$  

Of course, this is not the true solution, but the definition of the derivative tells us that the error is on the order of $\epsilon^2$.  

It is useful to note that instead of using the tangent line approximation and dropping the zeroth order terms, we could have obtained equation (2.25) by taking the derivative of both sides of the original equation

$$f(x_0 + \epsilon \bar{x}) = b_0 + \epsilon \bar{b}$$  

(2.26)  

with respect to $\epsilon$ and then setting $\epsilon = 0$. Doing this to the left side of (2.26) gives us

$$\frac{d}{d\epsilon} f(x_0 + \epsilon \bar{x})|_{\epsilon=0} = f'(x_0 + \epsilon \bar{x})|_{\epsilon=0} = f'(x_0)\bar{x}.$$  

Performing the same operations on the right side is trivial and yields the linear equation (2.25).  

We will use a generalization of this technique to get approximate solutions of the equation

$$ml\theta'' = -mg\sin\theta + f_\epsilon(\theta, \theta') - \epsilon g(t)$$  

(2.27)  

where $g : \mathbb{R} \to \mathbb{R}$ is a forcing function and $\epsilon$ is a small parameter. We will look for solutions close to stationary points, and we begin with $\theta_0 = 0$. We look for solutions of the form $\theta(t) = 0 + \epsilon \bar{\theta}(t)$. As above, we plug this into (2.27), take the derivative of both sides of the equation with respect to $\epsilon$ and then set $\epsilon = 0$. 


Here the right side is the more difficult.

\[
\left. \frac{d}{d\epsilon} \left( -mg \sin \epsilon \bar{\theta} + f_v(\epsilon \bar{\theta}, \epsilon \bar{\theta}') - \epsilon g(t) \right) \right|_{\epsilon = 0} = -mg(\cos \epsilon \bar{\theta})\bar{\theta} + \frac{\partial f_v}{\partial \theta}(\epsilon \bar{\theta}, \epsilon \bar{\theta}')\bar{\theta} + \frac{\partial f_v}{\partial \omega}(\epsilon \bar{\theta}, \epsilon \bar{\theta}')\bar{\theta}' - g(t) \right|_{\epsilon = 0} = -mg \cos \theta \bar{\theta} + \frac{\partial f_v}{\partial \theta}(0, 0)\bar{\theta} + \frac{\partial f_v}{\partial \omega}(0, 0)\bar{\theta}' + g(t) = -mg \bar{\theta} - \nu \bar{\theta}' + g(t).
\]

Here \( \nu := -\frac{\partial f_v}{\partial \omega}(0, 0) > 0 \) and we have used the fact that \( \frac{\partial f_v}{\partial \omega}(0, 0) = 0 \). (This last fact comes from the assumption that \( f_v(\theta, 0) \equiv 0 \).) Performing the same (but much easier) calculation on the left side yields the linearized equation

\[
ml\ddot{\bar{\theta}} = -mg \bar{\theta} - \nu \bar{\theta}' + g(t).
\]

This is a second order linear ODE with constant coefficients. The two-parameter family of solutions of the homogeneous \((g \equiv 0)\) problem is found by standard techniques which the reader should review. In the case where \( 4m^2\nu^2 > \nu^2 \) the solutions are given by

\[
\bar{\theta}(t) = e^{-\beta t}(A \cos \omega t + B \sin \omega t)
\]

where \( \beta := \frac{\nu}{2m}, \omega := \sqrt{\frac{\nu}{2m} - \beta^2} \), and \( A \) and \( B \) are arbitrary constants. Note that these solution always stay bounded. In fact, if \( \nu > 0 \) they decay to zero in time. The reader should check that this is true for all admissible cases of the parameter values.

We get a different result when we linearize about the stationary point \( \theta_0 = \pi \).

We now look for solutions of the form \( \theta(t) = \pi + \epsilon \bar{\theta}(t) \). As above, we plug this into (2.27), take the derivative of both sides of the equation with respect to \( \epsilon \) and then set \( \epsilon = 0 \). As before, the right side is the more difficult.

\[
\left. \frac{d}{d\epsilon} \left( -mg \sin(\pi + \epsilon \bar{\theta}) + f_v(\pi + \epsilon \bar{\theta}, \epsilon \bar{\theta}') - \epsilon g(t) \right) \right|_{\epsilon = 0} = -mg \cos(\pi + \epsilon \bar{\theta})\bar{\theta} + \frac{\partial f_v}{\partial \theta}(\pi + \epsilon \bar{\theta}, \epsilon \bar{\theta}')\bar{\theta} + \frac{\partial f_v}{\partial \omega}(\pi + \epsilon \bar{\theta}, \epsilon \bar{\theta}')\bar{\theta}' - g(t) \right|_{\epsilon = 0} = -mg \cos(\pi)\bar{\theta} + \frac{\partial f_v}{\partial \theta}(\pi, 0)\bar{\theta} + \frac{\partial f_v}{\partial \omega}(\pi, 0)\bar{\theta}' + g(t) = mg \bar{\theta} - \bar{\nu} \bar{\theta}' + g(t).
\]

Here \( \bar{\nu} := -\frac{\partial f_v}{\partial \omega}(\pi, 0) > 0 \) and we have used the fact that \( \frac{\partial f_v}{\partial \omega}(\pi, 0) = 0 \). Now our linearized ODE is

\[
ml\ddot{\bar{\theta}} = mg \bar{\theta} - \nu \bar{\theta}' + g(t).
\]

The change in sign on the coefficient of \( \bar{\theta} \) makes a huge difference in the character of solutions which are now given by

\[
\bar{\theta}(t) = Ae^{\beta^+ t} + Be^{\beta^- t}
\]
where
\[ \beta^\pm := \frac{1}{2ml}(-\dot{\nu} \pm \sqrt{\dot{\nu}^2 + 4m^2lg}). \]

Note that \( \beta^+ > 0 \) so there is a family of these solution that grows exponentially in time.

The situations above can be summarized in the following definition of stability.

**Definition 2.21.** Let \( \Omega \subset \mathbb{R}^n \) and let \( F : \Omega \to \mathbb{R}^n \) be uniformly Lipschitz. Let \( y_0 \in \Omega \) be a stationary point of the system of ODEs

\[ y' = F(y), \quad (2.28) \]

i.e. \( F(y_0) = 0 \). Then we say \( y_0 \) is **stable** if for every \( r > 0 \) there exists \( \delta > 0 \) such that if \( |y - y_0| < \delta \) then the solution of (2.28) with initial condition \( y(t_0) = \bar{y} \) exists on the time interval \([t_0, \infty)\) and

\[ y(t) \in B_r(y_0) \]

for all \( t > t_0 \). If \( y_0 \) is not stable we say it is **unstable**. If there exists \( \delta > 0 \) such that if \( |y - y_0| < \delta \) then the solution of (2.28) with initial condition \( y(t_0) = \bar{y} \) exists on the time interval \([t_0, \infty)\) and

\[ \lim_{t \to \infty} y(t) = y_0 \]

then we say \( y_0 \) is **asymptotically stable**.

**Remark 2.22.** The phase plane analysis of the previous section indicated that the stationary points corresponding to the pendulum hanging down were stable while those corresponding to the pendulum balanced on its pivot were unstable. However, it is much easier to prove these assertions about the linearized equations since we have closed form solutions. In light of this, we say that a stationary point is **linearly stable (unstable)** if the equations obtained by linearizing about that point are stable (unstable).