Intro. Sys. Nonlinear Eqs. / 1

\[ F : \mathbb{R}^n \to \mathbb{R}^n, \quad F(x) = 0 \]

\[ F(x) \text{ (exp of } x \text{ not solution)} : \text{(nonlinear) residual} \]

\[ x^* \text{ solution} \]

Methods generate sequence \( \{x_n\}_{n=0} \) iterates

\[ e = x - x^* \text{ error} \]

\[ \cdots \to F(x^*) \text{ residual} \]

\[ x^* \]

\[ F(x) \]

\[ \begin{pmatrix} p_1(x) \\ \vdots \\ p_n(x) \end{pmatrix} \to \begin{pmatrix} p_1(x_1, x_2, \ldots, x_n) \\ \vdots \\ p_n(x_1, x_2, \ldots, x_n) \end{pmatrix} \]

If all partial derivatives exist at \( x \), we call matrix \( F'(x) = \frac{\partial F}{\partial x} \) with

\[ F'(x)_{ij} = \frac{\partial p_i}{\partial x_j} = \frac{\partial (F)_i (x)}{\partial x_j} = f_{ij}(x) \]

the \textit{Jacobian} (of \( F \) at \( x \))

Given an iterate \( x_n \), we compute a new approximation using a 'local model' (typically linear)

\[ M_n(x) \approx F(x) \text{ in neighborhood of } x_n \]
This has 2 important consequences

1) We can solve the model equation (cheaply) but the solution will typically not be the exact solution of $F(x) = 0 \Rightarrow$ model not exact

This also means we may not want to solve the model to high accuracy.

Furthermore, the accuracy (error) of next iterate will depend on the accuracy of the model and the accuracy for which we solve model equation

2) The model is only locally valid

(and typically we have to estimate the "range" over which it is valid.)

Hence, if a step $x_{n+1} - x_n$ is too large, we cannot accept the step (in general) and we need additional modifications of the algorithm. (line search, damping, trust region, ...)

The standard local model for Newton is:

$$M_n(x) = F(x_n) + F'(x_n)(x-x_n)$$

Exact Newton's method (Newton step):

$$M_n(x_n) = 0 \Rightarrow F(x_n) + F'(x_n)(x-x_n) = 0 \Leftrightarrow$$

$$x-x_n = -\left(F'(x_n)^{-1} F(x_n) \Rightarrow$$

$$\text{Newton: } x_{n+1} = x_n - (F'(x_n))^{-1} F(x_n)$$
Convergence theory for Newton's method (local)

Local theory: \( x_0 \) "sufficiently" close to \( x^* \)

(we don't know how close this is)

Assumptions:
1. \( x^* \) exists
2. \( F' : \Omega \to \mathbb{R}^{n \times n} \) Lipschitz continuous near \( x^* \)
3. \( F'(x^*) \) is nonsingular

Lipschitz continuous near \( x^* \) →

\[
\| F'(x) - F'(y) \| \leq M \| x - y \|, \quad \forall x, y \in \mathcal{B}_e(x^*)
\]

(define vector 2-norm + matrix induced norm)

Theorem (given Assumptions): If \( x_0 \) sufficiently close to \( x^* \), then the seq of Newton iterates exists and converges to \( x^* \) such that

\[
\| e_{n+1} \| \leq K \| e_n \|^2
\]

for some \( K > 0 \) and \( n \) sufficiently large.

Proof: (outline)

\[
x_{n+1} = x_n - \beta_j(x_n)^{-1} F(x_n)
\]

\[
x_{n+1} - x^* = x_n - x^* - \beta_j(x_n)^{-1} (F(x_n) - F(x^*))
\]

\[
= \beta_j(x_n)^{-1} \left( F(x^*) - F(x_n) + \beta_j(x_n) (x_n - x^*) \right)
\]

\[
= \beta_j(x_n)^{-1} \left( F(x^*) - F(x_n) - \beta_j(x_n) (x^* - x_n) \right)
\]

(note = \( \beta_j(x_n)^{-1} (F(x^*) - M(x^*)) \))

\[
\| e_{n+1} \| = \| x_{n+1} - x^* \| = \| \beta_j(x_n)^{-1} (F(x^*) - F(x_n) - \beta_j(x_n) (x^* - x_n)) \|, \quad \text{bound}
\]
Consider \( \hat{F}(s) = F(x_n + s(x^* - x_n)) \)

\[
\hat{F}(1) = F(x^*) \quad \hat{F}(0) = F(x_n)
\]

Now components / coefficients \( \hat{F}(s) \) are \( \hat{F}_i : \mathbb{R} \to \mathbb{R} \) (so standard calculus)

\[
\hat{F}_i(s) = \frac{d}{ds} \hat{F}_i(0) = \int_0^1 \frac{d}{ds} \hat{F}_i(s) \, ds
\]

\[
\hat{F}_i(1) - \hat{F}_i(0) = \int_0^1 \frac{d}{ds} \hat{F}_i(s) \, ds
\]

\[
\hat{F}_i(s) = \sum_{j} \frac{\partial \hat{F}_i}{\partial (x^*_j)} (s) \cdot (x^*_j - x_n)_j
\]

\[
= \sum_{j} \frac{\partial \hat{F}_i}{\partial (x^*_j)} (x_n + s(x^* - x_n)) \cdot (x^*_j - x_n)_j
\]

\[
= \nabla \hat{F}_i(x_n + s(x^* - x_n))^T (x^* - x_n)
\]

\( i \)-th row of \( \hat{F}(s) \)

\[
\hat{F}(x^*) - F(x_n) = \hat{F}(1) - \hat{F}(0) = \int_0^1 (x_n + s(x^* - x_n))(x^* - x_n) \, ds
\]

\[
\hat{F}(x^*) - F(x_n) = \frac{\partial}{\partial y} \int_0^1 (x_n + s(x^* - x_n))(x^* - x_n) \, ds
\]

\[
= \int_0^1 \left[ \frac{\partial}{\partial y} (x_n + s(x^* - x_n)) - \frac{\partial}{\partial y} (x_n) \right] (x^* - x_n) \, ds
\]

\[
\| \hat{F}(x^*) - F(x_n) - \frac{\partial}{\partial y} (x_n)(x^* - x_n) \|^2 = \| \int_0^1 \left[ \frac{\partial}{\partial y} (x_n + s(x^* - x_n)) - \frac{\partial}{\partial y} (x_n) \right] (x^* - x_n) \, ds \|^2
\]

\[
\leq \int_0^1 \| \left[ \frac{\partial}{\partial y} (x_n + s(x^* - x_n)) - \frac{\partial}{\partial y} (x_n) \right] (x^* - x_n) \|^2 \, ds
\]

\[
\leq \int_0^1 \| y \| S(x^* - x_n) \| x^* - x_n \| \, ds \leq \| y \| \| x^* - x_n \| ^2 \int_0^1 S \, ds
\]

\[
= \frac{\chi}{2} \| x^* - x_n \|^2 = \frac{\chi}{2} \| e_n \|^2
\]
\[ \| e_{n+1} \| \leq \| g'(x_n)^{-1} \| \cdot \| F(x^*) - F(x_n) - g'(x_n)(x^* - x_n) \| \]
\[ \leq \frac{\tilde{K}}{2} \| e_n \|^2 \]

where \( \tilde{K} \geq \| g'(x) \| \) over neighborhood of \( x^* \).

Based on this result, do proof by induction.

**Issues:**

a) We assume \( \| g'(x^*)^{-1} \| \leq K^* \) (for some \( K^* > 0 \)).

Based on this and Lipschitz property need to show that \( \| g'(x_n) \| \) remains bounded over some region/neighborhood of \( x^* \).

b) We need to assure that all iterates \( x_n \) remain in neighborhood of \( x^* \) where \( g = F' \) exists,

\[ \| g'(x_n) \| \text{ bounded, Lipschitz property holds.} \]

c) We need to ensure a reduction of error, for example,

\[ \| e_{n+1} \| \leq \frac{1}{2} \| e_n \| \]

(at some point \( \| e_{n+1} \| \leq K \| e_n \|^2 \) will be stronger)

d) We assume open convex set \( D \subset \mathbb{R}^N \) such that \( F \) is differentiable in \( D \).

Take neighborhood \( N(x^*, r) = \{ x : \| x^* - x \| < r \} \subset D \)

such that \( g \in \text{Lip}_y(N(x^*, r)) \)

e) We have a lot of freedom in choosing norm we use, but preferably use corr./induced matrix norm.

This leads to requirement that \( x_0 \in N(x^*, \varepsilon) \) with

\[ \varepsilon = \min \left( \frac{1}{\beta K^*} \right) \]
To finish the proof, we need the following results:
(proof in finite dimensional setting is not hard and is left for homework)

Let $E \in \mathbb{R}^{N \times N}$ and $\|E\| < 1$ (where $\|\cdot\|$ is an induced norm and hence consistent, also implies $\|I\| = 1$)

Then $(I - E)$ is invertible ($(I - E)^{-1}$ exists) and

$$\| (I - E)^{-1} \| \leq \frac{1}{1 - \|E\|}$$

Let $A$ be invertible. Then if $B$ is sufficiently near $A$, $B$ is invertible too. More precisely

If $\| A^{-1} (B - A) \| < 1$, then

$B^{-1}$ exists and $\| B^{-1} \| \leq \frac{\| A^{-1} \|}{1 - \| A^{-1} (B - A) \|}$

We use this to prove that for $x$ sufficiently close to $x^*$, $g(x)^{-1}$ exists and we can bound $\| g(x)^{-1} \|$

There exists $\delta > 0$ such that $\forall x \in N(x^*, \delta)$:

$g(x)^{-1}$ exists and $\| g(x)^{-1} \| \leq 2\delta$

($\delta = \| g(x^*)^{-1} \|$)

Consider $\| g(x^*)^{-1} (g(x) - g(x^*)) \| \leq \| g(x^*)^{-1} \| \cdot \| g(x) - g(x^*) \|$

$\leq 3\delta \| x - x^* \|$ (using Lipschitz cont.)

If $x$ such that $\| x - x^* \| < \frac{1}{2}$ then using the above

$$\| g(x)^{-1} \| \leq \frac{\| g(x^*)^{-1} \|}{1 - \| g(x^*)^{-1} (g(x) - g(x^*)) \|} \leq \frac{\delta}{1 - \frac{1}{2}} = 2\delta$$
So, taking \( \frac{1}{\sqrt{2}} \| x - x^* \| < \frac{1}{2} \iff \| x - x^* \| < \frac{1}{2\sqrt{2}} \)

This gives \( \| g'(x) \| \leq 2\gamma \)

So, we can take \( \varepsilon = \gamma \)

Substitute this result in

\[
\| e_{n+1} \| \leq \| g'(x_n) \| \cdot \| F(x^*) - F(x_n) - g'(x_n)(x^* - x_n) \| \\
\leq 2\gamma \frac{\delta}{2} \| e_n \|^2 \\
= \gamma \frac{\delta}{2} \| e_n \|^2
\]

Now assume \( x_0 \in N(x^*, \varepsilon) \). Then

\[
\| e_0 \| = \| x_0 - x^* \| < \varepsilon \implies \\
\| e_i \| \leq \gamma \frac{\delta}{2} \| e_0 \| \cdot \| e_0 \| \\
\leq \gamma \frac{\delta}{2} \varepsilon \| e_0 \| = \frac{1}{2} \| e_0 \| \quad (\text{since } \varepsilon = \gamma \frac{1}{2\sqrt{2}} \iff \gamma \varepsilon = \frac{1}{2})
\]

So, \( x_i \in N(x^*, \varepsilon) \) and we can prove convergence by induction (linear converge at least).

We assume \( F \) continuously differentiable in \( N(x^*, r) \). We must take region where all conditions hold.

So, \( x_0 \in N(x^*, \eta) \) where \( \eta = \min(5\gamma \frac{1}{2\sqrt{2}}) \)
Convergence \( \| e_{n+1} \| \leq K \| e_n \|^2 \) is called *\( q \)-quadratic convergence*.

(Overview of types of convergence)

- Number of significant digits in answer roughly doubles each iteration.

(Unfortunately, we don’t know error)

- Of \( F'(x^*) \) not ill-conditioned we see quadratic reduction of residual.

\[
F(x_{n+1}) = F(x^*) + \frac{1}{2} J(x^*)(x_{n+1} - x^*)
\]

\[
\| F(x_{n+1}) \| \leq \| J(x^*) \| \| e_{n+1} \|
\]

\( \Rightarrow \) bounded by constant \( K \) if

- \( J(x^*) \) ill-cond then \( \| J(x^*) p \| \) may depend strongly on ‘direction’ of \( p \).
Steps of Newton algorithm

1) evaluate $F(x_n)$ and test for convergence

2) approximate solution of $F'(x_n)s = -F(x_n)$
   
   or more generally $M_n(x) = 0$ or $\|M_n(x)\| \leq \text{tol}$
   
   (approximate solution local model)

3) $x_{n+1} = x_n + ds$, where step length is selected to
   guarantee (sufficient) decrease of $\|F\|$  
   (in addition step should be small, not too small)

Typically (2) is expensive part, but (3) can
be expensive if function eval is expensive (and
needs to be done multiple times)

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Show 10 examples

1) $\arctan(x) = 0$

2) $\tan(x) - x = 0$

discuss:

a) Need line search when far from solution, more
   general issue of ill-cond. "jac [scalar $F' \approx 0$]
   and limitations of local model.
   (stepsize control)

   ill-cond $\rightarrow$ directions in which in rel. sense $F$
   changes very slowly $\rightarrow$ geometric notions
b) Accuracy we can obtain bounded by accuracy in function evaluation and accuracy in derivative evaluation

Note Newton step is "residual correction"

$$\tilde{x}_{nn} = \frac{1}{y_n}(F(x_n) + \tilde{e}_n)$$

$$x_{nn} = x_n - \tilde{e}_n = x_n - \frac{1}{y_n}F_n - \frac{1}{y_n}e_n$$

and on acc. in solution based on $\|y(x)\|_{\infty}$

(will also show up in convergence of $\|F_n\|$

(not on acc. Jac. of $\|y\|_{\infty} < c\|F_n\|$)

In general acc in $F(x)$ not issue: conver will stagnate at some point, but typ. suff. accurate.

We may not want to compute exact $F'(x_n)$ (cost)

We may not want to solve model eq. exactly

Convergence result for Newton with inexact $F'$ and inexact $F$.

Theorem 1.2 (Std. Assumptions)

matrix $A(x): \|A(x)\| < \delta_y$

vector $e(x): \|e(x)\| < \delta_f$

For all $x$ near $x^*$ (neighborhood)

If $x_0$ suff. near $x^*$ and $\delta_y, \delta_f$ suff. small then

$$x_{nn} = x_n - (F'(x_n) + A(x_n))^{-1}(F(x_n) + e(x_n))$$

Well-defined $(F'(x_n) + A(x_n))^{-1}$ exists (bounded)

and
\[ \| e_{n+1} \| \leq K (\| e_n \|^2 + \| A(x_n) \| \| e_n \| + \| e(x_n) \|) \]

for some \( K > 0 \)

Proof: see Kelley, Iterative Methods for Linear and Nonlinear Equations (TMLNE)

(proof follows same lines as "exact" version with some extra bounds)

Note that errors in Jac. eval. lead, in general, to linear convergence.

If \( \| A(x_n) \| \leq c \| e_n \| \) and \( e(x_n) \) sufficiently small, we retain quadratic convergence.

As Newton's method can be expensive we often look for ways to reduce cost.

Main costs (usually): 
- i) compute \( F'(x_n) \) \( (N^2) \)
- ii) solve for \( F'(x_n) \) \( (N^3) \)

\( \rightarrow \) Approximate Jacobian.

* especially efficient if approx. also reduces cost of solving for Jacobian

\( \star \) balance slower convergence with (significantly) lower cost per iteration.

1) Chord method / modified Newton

replace \( F'(x_n) \) by \( F'(x_0) \) (compute once, factorize once)

If \( x_0 \) close enough to \( x^* \) then convergence is quadratic: \( \| e_{n+1} \| \leq p \| e_n \| \) \( (p \in [0,1)) \)

for \( n \) suff. large.
Using Theo 1.2: \((F'(x_n) + F'(x_0) - F'(x_n))^T \cdots\)

So \(\Delta(x_n) = F'(x_0) - F'(x_n) \Rightarrow\)

\[ ||\Delta(x_n)|| = ||F'(x_0) - F'(x_n)|| \leq 2y ||x_0 - x_n||\]

Assuming criteria satisfied and hence convergence (monotone \(\Rightarrow\) reduction each step)

\[ ||\Delta(x_n)|| \leq y ||x_0 - x_n|| = O(||x_0||)\]

If we take \(N_r(x^*)\) where convergence guaranteed and (obviously \(x_0 \in N_r(x^*)\)) \(\Rightarrow\)

\[ ||\Delta(x_n)|| \leq y ||x_0 - x_n|| \leq 2y ||x_0 - x^*|| = 2y ||e_0||\]

Taking \(\varepsilon(x_n) = 0\) or suff. small (usually machine \(\leq\) negligible precision)

\[ ||e_{n+1}|| \leq k(||e_n||^2 + c ||e_0|| ||e_n|| + \varepsilon)\]

\[ \leq \frac{a}{p} ||e_n|| \quad \text{where if} \ ||e_n|| \text{ suff} \text{ small and} \ ||e_0|| \to 0 \]

\(p\) is called \(q\)-factor

For \(q\)-linear convergence typically

\[ ||e_{n+1}|| \approx p ||e_n|| \] and \[ ||F(x_{n+1})|| \approx p ||F(x_n)||\]

\[ 2) \text{Secant method} \]

\[ 1D: \ x_{n+1} = x_n - \frac{F(x_n)}{F(x_n) - F(x_{n-1})} \Rightarrow \]

\[ x_{n+1} = x_n - \frac{F(x_n)(x_n - x_{n-1})}{F(x_n) - F(x_{n-1})} \]
Backward difference approx. for \( F'(x_n) = \frac{F(x_n) - F(x_{n-1})}{x_n - x_{n-1}} \)

\[
F(x_n) - F(x_{n-1}) = \int_{x_{n-1}}^{x_n} F'(s) \, ds
\]

\[
\left| \int_{x_{n-1}}^{x_n} F'(s) \, ds - F'(m)(x_n - x_{n-1}) \right| = \int_{x_{n-1}}^{x_n} F'(s) - F'(m) \, ds
\]

\[
\leq \int_{x_{n-1}}^{x_n} |F'(s) - F'(m)| \, ds \leq \int_{x_{n-1}}^{x_n} |s - m| \, ds = \frac{1}{2} \int_{x_{n-1}}^{x_n} (m-s)^2 \, ds
\]

\[
= \frac{1}{4} (x_n - x_{n-1})^2 + \frac{1}{2} \int_{m}^{x_n} (s - m) \, ds
\]

or Taylor \( F(x_{n+1}) = F(x_n) + F'(s)(x_n - x_{n-1}) \) for \( s \in (x_n, x_{n+1}) \)

\[
x_{n+1} = x_n - \frac{F(x_n)(x_n - x_{n-1})}{F'(s)(x_n - x_{n-1})} \text{ for } s \in (x_n, x_{n+1}) \Rightarrow \| \Delta x_n \| \leq \| \Delta x_{n-1} \|
\]

\[
\frac{|\Delta x_n|}{|\Delta x_{n-1}|} \leq K \left( |\Delta x_{n-1}|^2 + |\Delta x_{n-1}|^3 + \varepsilon \right) \Rightarrow
\]

\[
\frac{|\Delta x_n|}{|\Delta x_{n-1}|} \leq K (|\Delta x_{n-1}| + |\Delta x_{n-1}|) \leq 2K |\Delta x_{n-1}| \rightarrow 0
\]

q - superlinearly convergent

There is no "obvious" extension to higher dimensions but many methods have been proposed.

Most well-known: Broyden's method (more than one)

\( \rightarrow \text{BFGS, } \ldots \)

Basic idea: update (approximate) Jacobian by low rank update satisfying certain conditions
Another way to reduce costs are

Inexact Newton methods → solve Newton step inexactly

\[ F'(x_n) s = -F(x_n) \]

by
\[ \| F'(x_n) s + F(x_n) \| \leq \eta \| F(x_n) \| \]

where \( s = x_{n+1} - x_n \) (Newton step)

Forcing term \( \eta \) typically varied during iteration

- \( \eta \) small → close to Newton but expensive
- \( \eta \) large → slower convergence "cheaper"

Far from solution Newton converges linear at best

\[ \text{sequence } \eta_n \rightarrow 0 \]

Cheap far from solution

Recover (nearby) quadratic convergence close to solution

**Theo 1.3 (Std Assumptions)**

There are \( s \) and \( \eta \) such that, if \( x_0 \in B(s) \),

\[ \{ \eta_n \} \subset [0, \eta], \text{ then inexact Newton iteration} \]

\[ x_{n+1} = x_n + s_n, \text{ where} \]

\[ \| F'(x_n) s_n + F(x_n) \| \leq \eta_n \| F(x_n) \| \]

converges \( q \)-linearly to \( x^* \)

Moreover,
* If $\eta_n \to 0$, convergence is $q$-superlinear
* If $\eta_n \leq K_n \| F(x_n) \|^p$ for some $K_n > 0$, convergence is $q$-superlinear with $q$-order $1+p$

Proof: (take $\Delta(x_n) = 0$) second part

$F'(x_n)^T \delta_n = -F(x_n) + \hat{e}(x_n)$, where $\| \hat{e}(x_n) \| \leq \eta_n \| F(x_n) \|

Corresponds to incorrect function eval plus exact solve

$\| F(x_n) \| = \| F(x^*) \| + \int_0^1 F'(x^* + t(x_n - x^*)) (x_n - x^*) \, dt$

$= F(x^*) + F'(x^*) (x_n - x^*) + \hat{e}$

$\| F(x_n) \| \leq \| F'(x^*) \| \cdot \| e_n \| + \frac{1}{2} \| e_n \|^2$

$= \| e_n \| \left( \| F'(x^*) \| + \frac{1}{2} \| e_n \| \right)$

Theo 1.2 $\Rightarrow \| e_n \| \leq K \left( \| e_n \|^2 + \eta_n \| e_n \| \left( \| F'(x^*) \| + \frac{1}{2} \| e_n \| \right) \right)$

$\eta_n \leq K_n \| F(x_n) \|^p \leq K_n \| e_n \|^p \left( \| F'(x^*) \| + \frac{1}{2} \| e_n \| \right)^p$

$\| e_n \| \leq K \left( \| e_n \|^2 + \eta_n \| e_n \|^p \left( \| F'(x^*) \| + \frac{1}{2} \| e_n \| \right)^p \right)$

Convergence is $q$-superlinear with $q$-order $1+p \leq 2$
Termination criterion: \( \| F(x^*) \| \leq T \| F(x_0) \| + \tau \)

(as before) \( N(x^*, \varepsilon) \) such that \( B(\varepsilon) \) in book

\[ x \in N(x^*, \varepsilon) : \| F'(x) \| \leq 2 \| F'(x^*) \| \quad \text{and} \quad \| (F'(x))^{-1} \| \leq 2 \| (F'(x^*))^{-1} \| \]

Then \( \| F'(x^*)^{-1} \| \| e \|_2 \leq \| F(x) \| \leq 2 \| F'(x^*) \| \cdot \| e \|_2 \)

Also holds for \( F(x_0) \to \)

\[ \frac{1}{2} \| F'(x^*)^{-1} \| \| e \|_2 \leq \| F(x) \| \leq \frac{2 \| F'(x^*) \| \cdot \| e \|_2}{2 \| F(x^*) \| \| e_0 \|} \iff \]

\[ \frac{1}{4 x(F'(x^*))^{-1} \cdot \| e \|_2} \leq \frac{\| F(x) \|}{\| F(x^*) \|} \leq \frac{4 x(F'(x^*))^{-1} \cdot \| e \|_2}{\| F(x^*) \| \| e_0 \|} \]

relative reduction of residual norm bounded from above and below by constant times relative reduction of error norm.

If superlinear convergence \( \left( \frac{\| e_{n+1} \|}{\| e_n \|} \to 0 \right) \)

\[ e_{n+1} = e_n + s_n \iff x_{n+1} - x^* = x_n - x^* + s_n \]

\[ \| e_{n+1} \| = o'(\| e_n \|) \]

Hence \( s_n = -e_n + o(\| e_n \|) \Rightarrow \| s_n \| \sim \| e_n \| \)

(current) rate of convergence: \( p_n = \frac{\| s_{n+1} \|}{\| s_n \|} \sim \frac{\| e_{n+1} \|}{\| e_n \|} \geq \frac{\| e_{n+1} \|}{\| e_n \|} \)

(for \( n \) suff. large)

So, for \( n \) suff. large:

\[ \| e_{n+1} \| \leq p_n \| e_n \| \sim \frac{\| s_{n+1} \|^2}{\| s_n \|} \]

For superlin conv. iteration
(stopping crit) \[ \| s_n \|_2 / \| s_{n-1} \| < \tau \]

implies that \[ \| e_{n+1} \| < \tau \]

What for linearly convergent process

\[ p = \| s_n \| / \| s_{n-1} \| \quad \text{or} \quad p = \left( \frac{\| s_n \|}{\| s_{n-1} \|} \right)^\frac{1}{n} \]

\[ e_n = e_{n-1} - s_n \Rightarrow \| e_n \| \leq \| e_{n-1} \| + \| s_n \| \Rightarrow \]
\[ \| e_n \| - \| s_n \| \leq \| e_{n-1} \| = p \| e_n \| \ (K) \]
\[ \| e_n \| / p = \| e_n \| / \| e_{n-1} \| \leq \frac{\| s_n \|}{\tau i - p} \]

Terminate when \[ \| s_n \| < \frac{\tau (1 - p)}{p} \], then if \( p \) is overestimate

\[ \| e_{n+1} \| \leq p \| s_n \| / (1 - p) \leq \tau \]

(in practice use an additional safety factor)

\[ (\ast 1) \quad \| s_n \| \leq \| e_n \| / p - \| e_n \| = \| e_n \| (p - 1) \]
\[ \| s_n \| \leq \| e_n \| (1 - p) \Rightarrow \| e_n \| = \| s_n \| \frac{1}{1 - \tau p} \]
\[ \| e_{n+1} \| \leq \| s_n \| / \left( \frac{1}{1 - \tau p} \right) \]
\[ \| e_{n+1} \| = p \| e_n \| \leq \| s_n \| / \left( \frac{p}{1 - \tau p} \right) \]
If local model violated $\Rightarrow$ Newton step too large, we cannot trust Newton's step (solution) to be accurate, or even $F$ to be defined.

Extreme case $F'(x_n)$ is singular $\Rightarrow \|F'(x_n)^{-1}\| = \infty$

Line search (damped Newton) to reduce step size

$$d_n = -(F'(x_n))^{-1}F(x_n), \quad s = d d_n$$

1D equation $L(d) = \|F(x_n + d d_n)\|^2$

Sufficient decrease (Armijo Rule):

$$\|F(x_n + 2^{-m}d_n)\| \leq (1-\alpha 2^{-m}) \|F(x_n)\| \text{ for } \alpha \in (0, 1)$$

Typical value $\alpha = 10^{-4}$

Accept smallest $m > 0$ that satisfies rule

In some cases not aggressive enough, many line search steps $\Rightarrow$ too expensive

Eval. of $F(x_n + dd_n)$ can be quite expensive

Note that for $m = 1$ we have 3 values for $L(d)$:

$L(0), L(1/2), L(1) \Rightarrow$ use for poly. approx. and minimize polynomial (cheap!) subject to reasonable decrease of stepsize: $p(d)$ sake $L(0), L(dm), L(dm_{-1})$; find

$$\min \{p(d) : d \in \left[\frac{d_m}{2}, \frac{d_m}{10}\right]\}$$

Line search only robust if $F(x_n)$ accurate. So, if line search not effective and using approx. $Jac.$ $\Rightarrow$

Compute (more) accurate Jacobian.
Linesearch ends for smallest $m > 0$ such that

$$\| F(x_n + d_m d) \| \leq (1 - \alpha d_m) \| F(x_n) \|$$

**Basic Newton Algorithm**

**Input** $x$ (initial guess)
- $F$ function handle/pointer
- $t_a$ absolute tolerance
- $t_r$ relative tolerance

**Eval. $F(x)$;** $t = t_r \| F(x) \| + t_a$

while $\| F(x) \| > t$ do

"Solve" for $d$ s.t. $\| F(x) + F'(x)d \| \leq \eta \| F(x) \|$

(terminate with failure if not successful)

$d = 1$

while $\| F(x + dd) \| \geq (1 - \alpha d) \| F(x) \|$ do

$d = \sigma d$, where $\sigma \in [\frac{1}{10}, \frac{1}{5}]$ minimizing

polynomial model of $\| F(x + dd) \|^2$

end while

$x = x + dd$
end while

One of 3 possibilities:

i) $\{x_n\} \rightarrow x^*$ where std. assump. hold

ii) $\{x_n\}$ is unbounded

iii) $F(x_n)$ will become singular

See Theo: 1.4

Read remainder of chapter (1.7.1 pp)