The Calculus of Residues

The Residue Theorem  The Laurent series for a function \( f(z) \) analytic in a region \( \mathcal{R} \) which includes \( z \) such that \( 0 < |z - z_0| \leq R \) takes the form

\[
f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k, \quad a_k = \frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{f(w)}{(w - z_0)^{k+1}} dw, \quad 0 < r < R,
\]

where \( \mathcal{C}_r \) is a circular contour of radius \( r \) centered at \( z_0 \). Our interest centers on the value \( k = -1 \). For this value of \( k \) we have

\[
a_{-1} = \frac{1}{2\pi i} \int_{\mathcal{C}_r} f(w) \, dw \Rightarrow \int_{\mathcal{C}_r} f(w) \, dw = 2\pi i a_{-1}.
\]

The coefficient \( a_{-1} \) is called the residue of \( f(z) \) at the point \( z_0 \); we often write it in the form

\[
a_{-1} = \text{Res} f(z) \bigg|_{z_0}.
\]

Now suppose \( f(z) \) is analytic in a region \( \mathcal{R} \) which includes a simple closed curve \( \mathcal{C} \) and the interior of \( \mathcal{C} \), except for a finite number of points \( z_k, \, k = 1, 2, ..., n \), where \( f(z) \) has isolated singularities. About the points \( z_k \) we form circles \( \mathcal{C}_k \) of radii \( r_k > 0 \) chosen so that the \( \mathcal{C}_k \) all lie in the interior of \( \mathcal{C} \) and do not intersect each other (this last condition is simply a convenience, not a necessity). Applying the extended version of Cauchy’s theorem we then have

Theorem 1 (Residue Theorem)  Under the above assumptions,

\[
\left( \int_{\mathcal{C}} - \sum_{k=1}^{n} \int_{\mathcal{C}_k} \right) f(z) \, dz = 0
\]

\[
\Rightarrow \int_{\mathcal{C}} f(z) \, dz = \sum_{k=1}^{n} \int_{\mathcal{C}_k} f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res} f(z) \bigg|_{z_k}.
\]

We should emphasize here that each \( z_k \) is a separate isolated singularity of \( f(z) \) in the interior of \( \mathcal{C} \); about each \( z_k \) the function \( f(z) \) has a Laurent series, depending on \( f(z) \) and the point \( z_k \),

\[
f(z) = \sum_{j=-\infty}^{\infty} a_{k,j} (z - z_k)^j, \quad a_{k,j} = \frac{1}{2\pi i} \int_{\mathcal{C}_k} \frac{f(w)}{(w - z_k)^{j+1}} dw,
\]
convergent in a “punctured disk” \( 0 < |z - z_k| < R_k \), where \( R_k \) is the minimum distance from \( z_k \) to the other \( z_\ell \) and the boundary of \( \mathcal{R} \). In particular, then, \( \text{Res} f(z) \big|_{z_k} = a_{k,-1} \).

### Evaluation of Residues

The Residue Theorem forms the basis for a powerful technique allowing us to evaluate a large number of important definite integrals and to sum certain infinite series. Before we can use it in this way, however, we need more effective ways to identify residues, i.e., the coefficients \( a_{-1} \) in Laurent series \( f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \). The formula \( a_{-1} = \frac{1}{2\pi i} \int_{C_r} f(z) \, dz \) is generally not very useful in this respect; in fact it more or less begs the question because it is usually the integral we are trying to evaluate. One way in which \( a_{-1} \) can be evaluated is to compute the Laurent series by manipulation of power series, as illustrated in the Laurent series section. However, this method is often rather cumbersome; the following procedure normally works better if the isolated singularity is a pole of \( f(z) \), which is typically the case in applications.

Suppose \( f(z) \) has a pole of order \( n \) at \( z_0 \), so that

\[
f(z) = a_{-n} (z-z_0)^{-n} + \cdots + a_{-1} (z-z_0)^{-1} + a_0 + a_1 (z-z_0) + a_2 (z-z_0)^2 + \cdots.
\]

We multiply by \( (z-z_0)^n \) to obtain

\[
(z-z_0)^n f(z) = a_{-n} + \cdots + a_{-1} (z-z_0)^{n-1} + a_0 (z-z_0)^n + a_1 (z-z_0)^{n+1} + a_2 (z-z_0)^{n+2} + \cdots
\]

and then differentiate \( n - 1 \) times to get

\[
\frac{d^{n-1}}{dz_0^{n-1}} ((z-z_0)^n f(z)) = (n-1)! a_{-1} + n! a_0 (z-z_0) + (n+1)n(n-1) \cdots 3 \cdot 2 a_1 (z-z_0)^2 + \cdots.
\]

Then taking the limit as \( z \to z_0 \) we have

\[
\lim_{z \to z_0} \frac{d^{n-1}}{dz_0^{n-1}} ((z-z_0)^n f(z)) = (n-1)! a_{-1}.
\]

Solving for \( a_{-1} \) we have the formula

\[
a_{-1} = \text{Res} f(z) \big|_{z_0} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz_0^{n-1}} ((z-z_0)^n f(z)).
\]
In the case of a simple pole, where $n = 1$, we have the simpler formula

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z).$$

**Example 1** We find the residue of $f(z) = \frac{\sin(z)}{z^2 - 1}$ at $z = 1$. Since $z^2 - 1 = (z - 1)(z + 1)$ we expect a first order pole at $z = 1$. Clearly we have

$$\frac{\sin(z)}{z^2 - 1} = \frac{\sin(z - 1 + 1)}{(z - 1)(z + 1)} = \frac{\sin(z - 1) \cos(1) + \cos(z - 1) \sin(1)}{(z - 1)(z + 1)}.$$

Then

$$\lim_{z \to 1} (z - 1) \frac{\sin(z)}{z^2 - 1} = \lim_{z \to 1} \frac{\sin(0) \cos(1) + \cos(0) \sin(1)}{2} = \frac{\sin(1)}{2}$$

and we have $a_{-1} = \frac{\sin(1)}{2}$ as the residue at $z = 1$. A similar computation shows that the residue at $z = -1$ is also equal to $\frac{\sin(1)}{2}$. As a consequence, if $C_r$ is a circle of radius $r > 1$ centered at $z = 0$ we have

$$\int_{C_r} f(z) \, dz = 2\pi i \sin(1).$$

**Example 2** Let $g(z) = \frac{e^z}{\sin^2 z}$ and let $z_0 = 0$. Since $\frac{\sin z}{z} \to 1$ as $z \to 0$, we expect to have a pole of order 2 at 0. Thus we have

$$a_{-1} = \frac{1}{1!} \lim_{z \to 0} \left( \frac{d}{dz} \left( \frac{z^2 e^z}{\sin^2 z} \right) \right) = \lim_{z \to 0} \left( \frac{2z e^z + z^2 e^z}{\sin^2 z} - \frac{2z^2 e^z \cos z}{\sin^3 z} \right)$$

$$= \lim_{z \to 0} \left( \frac{z^2 e^z}{\sin^2 z} + 2e^z - \frac{z^2 e^z \cos z}{\sin^3 z} \right)$$

$$= 1 + 2 \left( \lim_{z \to 0} \frac{z e^z}{\sin z} \right) \lim_{z \to 0} \left( \frac{\sin z - z \cos z}{\sin^2 z} \right) = 1 + 2 \lim_{z \to 0} \left( \frac{\sin z - z \cos z}{\sin^2 z} \right).$$

Applying l’Hôpital’s rule to the limit we find that

$$\lim_{z \to 0} \left( \frac{\sin z - z \cos z}{\sin^2 z} \right) = \lim_{z \to 0} \left( \frac{\cos z - \cos z + z \sin z}{2 \sin z \cos z} \right) = \lim_{z \to 0} \left( \frac{z}{2 \cos z} \right) = 0.$$

We conclude that $a_{-1} = 1$ in this example.
Evaluation of Certain Trigonometric Integrals

One of the most direct applications of the Residue Theorem enables us to calculate certain integrals of the form

\[ \int_0^{2\pi} K(\cos \theta, \sin \theta) \, d\theta, \]

with certain assumptions on the expression \( K(\cos \theta, \sin \theta) \). On the unit circle \( |z| = 1 \) in the complex plane we have \( z = \cos \theta + i \sin \theta \), \( \frac{1}{z} = \overline{z} = \cos \theta - i \sin \theta \). Solving these two equations for \( \cos \theta \) and \( \sin \theta \) we obtain

\[ \cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right), \quad \sin \theta = \frac{1}{2i} \left( z - \frac{1}{z} \right). \]

Thus, for \( z = \cos \theta + i \sin \theta \) we have

\[ K(\cos \theta, \sin \theta) = K \left( \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2i} \left( z - \frac{1}{z} \right) \right). \]

Further, we have

\[ dz = d \left( re^{i\theta} \right) = ire^{i\theta} \, d\theta = iz \, d\theta \quad \Rightarrow \quad d\theta = \frac{dz}{iz} = -\frac{dz}{z}. \]

The required assumption on \( K \) is that \( K \left( \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2i} \left( z - \frac{1}{z} \right) \right) \) should extend as an analytic function of \( z \) to a region which includes the unit circle and its interior, except for isolated singularities, none of them on the unit circle and only finitely many of them in the interior of that circle. With \( C \) denoting the unit circle, we then have

\[ \int_0^{2\pi} K(\cos \theta, \sin \theta) \, d\theta = -i \int_C K \left( \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2i} \left( z - \frac{1}{z} \right) \right) \frac{dz}{z}, \]

\[ = 2\pi \sum_{k=1}^{n} \text{Res} \left( \frac{K \left( \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2i} \left( z - \frac{1}{z} \right) \right) \frac{dz}{z} \right) \bigg|_{z_k}, \]

where \( z_k, \ k = 1, 2, ..., n \), are the isolated singularities of \( \frac{K \left( \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2i} \left( z - \frac{1}{z} \right) \right) \frac{dz}{z} \) lying in the interior of the unit circle.

**Example 3** We compute \( \int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} \). Here we have

\[ \frac{K \left( \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2i} \left( z - \frac{1}{z} \right) \right)}{z} = \frac{1}{2z + \frac{1}{2} \left( z + \frac{1}{z} \right)} \]

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\[
\frac{1}{\frac{x^2}{2} + 2z + \frac{1}{2}} = \frac{2}{z^2 + 4z + 1} = \frac{2}{(z + 2 - \sqrt{2})(z + 2 + \sqrt{2})}.
\]

Only the isolated singularity \(\sqrt{2} - 2\) lies inside the unit circle. There the residue may be computed as
\[
\lim_{z \to \sqrt{2} - 2} \left( \frac{z + 2 - \sqrt{2}}{(z + 2 - \sqrt{2})(z + 2 + \sqrt{2})} \right) = \frac{2}{z + 2 + \sqrt{2}} = \frac{1}{\sqrt{2}}.
\]

Accordingly, we conclude that \(\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \frac{2\pi}{\sqrt{2}}\).

**Example 4** We compute \(\int_0^{2\pi} \frac{d\theta}{1 + \sin^2 \theta}\). In this case
\[
\frac{K \left( \left( \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2z} \left( z - \frac{1}{z} \right) \right) \right)}{z} = \frac{1}{1 + \left( \frac{1}{2z} \left( z - \frac{1}{z} \right) \right)^2} \frac{z}{z} = \frac{-z}{\left( 1 - \frac{1}{4} \left( z + \frac{1}{z} \right) \right)^2} z = \frac{-z}{\left( z - \frac{1}{2} z^2 + \frac{1}{4} \right) \left( z^2 - 3 + 2\sqrt{2} \right)} = \frac{-z}{\left( z + \sqrt{3 - 2\sqrt{2}} \right) \left( z - \sqrt{3 - 2\sqrt{2}} \right) \left( z^2 - 3 - 2\sqrt{2} \right)}.
\]

Here we have two first order poles inside the unit circle, at \(z = \pm \sqrt{3 - 2\sqrt{2}}\). The residue at \(z = \sqrt{3 - 2\sqrt{2}}\) is
\[
\lim_{z \to \sqrt{3 - 2\sqrt{2}}} \frac{-4z \left( z - \sqrt{3 - 2\sqrt{2}} \right)}{(z + \sqrt{3 - 2\sqrt{2}}) \left( z - \sqrt{3 - 2\sqrt{2}} \right) \left( z^2 - 3 - 2\sqrt{2} \right)} = \frac{-4z}{\left( z + \sqrt{3 - 2\sqrt{2}} \right) \left( z - \sqrt{3 - 2\sqrt{2}} \right) \left( z^2 - 3 - 2\sqrt{2} \right)} = \frac{1}{2\sqrt{2}}.
\]

Much the same computation shows that the residue at \(z = -\sqrt{3 - 2\sqrt{2}}\) has the same value and we conclude that
\[
\int_0^{2\pi} \frac{d\theta}{1 + \sin^2 \theta} = 2\pi \left( \frac{1}{\sqrt{2}} \right) = \sqrt{2}\pi.
\]