Integrals and their Approximation

Integrals over Regions in the Plane Let $\mathcal{R}$ be a region in $\mathbf{R}^2$ and let $\phi(X)$ be a continuous, or piecewise continuous, function defined on $\mathcal{R}$. For the purposes of this exposition we will assume the region $\mathcal{R}$ is bounded; we will remark later on the situation where this is not the case.

To define $\int_{\mathcal{R}} \phi(X)\, dX$ we proceed in a manner which is an extension of the familiar procedure for one dimensional integrals. We define a *rectangular partition* of $\mathcal{R}$ to be a finite set of non-overlapping rectangles, $\{ R_k \,|\, k = 1, 2, ..., K \}$, in the plane $\mathbf{R}^2$, such that the R_k *cover* $\mathcal{R}$, i.e.,

$$\mathcal{R} \subset \cup_{k=1}^{K} R_k,$$

and all of the R_k are necessary in the sense that if we exclude one of them, say R_j, then

$$\mathcal{R} \not\subset \cup_{k=1,...,K,\, k\neq j} R_k.$$

We could say that the R_k *just cover* $\mathcal{R}$. In particular, then, for each k the intersection $R_k \cap \mathcal{R}$ is not the empty set.

We then form a finite sum

$$\sum_{k=1}^{K} \phi(X_k)\mathcal{A}(R_k),$$

where $\mathcal{A}(R_k)$ denotes the area of the rectangle R_k and X_k is any point in $R_k \cap \mathcal{R}$, $k = 1, 2, ..., K$. Defining $d(R_k)$, the *diameter* of R_k, by

$$d(R_k) = \sup \left\{ \left\| X - \hat{X} \right\| \,|\, X, \hat{X} \in R_k \right\},$$

we let

$$h = max_{k=1,...,K} \{d(R_k)\}.$$

It can then be shown that

$$\lim_{h \to 0} \sum_{k=1}^{K} \phi(X_k)\mathcal{A}(R_k)$$

exists; there is a number $\mathcal{I}$ such that, as $h \to 0$, the above sum approaches $\mathcal{I}$ as $h \to 0$, no matter how the partition is formed and no matter what choices we make for the sample points X_k where $\phi(X)$ is evaluated. That number, $\mathcal{I}$, is called the *integral of* $\phi(X)$ *over* $\mathcal{R}$, and is denoted by $\int_{\mathcal{R}} \phi(X)\, dX$. It has the familiar linearity property

$$\int_{\mathcal{R}} (\alpha\, \phi(X) + \beta\, \psi(X))\, dX \;=\; \alpha \int_{\mathcal{R}} \phi(X)\, dX \;+\; \beta \int_{\mathcal{R}} \psi(X)\, dX$$

for two functions $\phi(X)$, $\psi(X)$ as described, and two scalars α, β. Further, the integral is *additive* in the following sense: if the region $\mathcal{R}$ is formed as the union of two disjoint, i.e., non-overlapping, regions $\mathcal{R}_1$ and $\mathcal{R}_2$, then

$$\int_{\mathcal{R}} \phi(X)\, dX \;=\; \int_{\mathcal{R}_1} \phi(X)\, dX \;+\; \int_{\mathcal{R}_2} \phi(X)\, dX.$$

In reality it would not be necessary to use rectangles in the partition of the region $\mathcal{R}$. Any sets with well defined area and diameter, such as triangles, parallelograms, hexagons, curvilinear rectangles, triangles and parallelograms, etc., which can be "tiled" together to cover the region $\mathcal{R}$ in the manner described above for rectangles could be used. In fact, as we will see, triangular partitions, with only slightly modified requirements from those imposed in the foregoing, form the basis for the *finite element* approach used extensively in engineering applications.

The Trapezoidal Rule for Two-Dimensional Integrals

We will begin with the simple case wherein $\mathcal{R}$ is, itself, a rectangle in $\mathbf{R}^2$:

$$\mathcal{R} \;=\; \left\{\, (x,y) \,\in\, \mathbf{R}^2 \,|\, a \,\leq\, x \,\leq\, b,\;\; c \,\leq\, y \,\leq\, d \,\right\}.$$

We introduce partitions of the intervals $a \le x \le b$ and $c \le y \le d$ consisting of $N+1$ and $M+1$ points, respectively, where N and M are positive integers, not necessarily equal:

$$a = x_0 < x_1 < \cdots < x_{k-1} < x_k < x_{k+1} < \cdots < x_{N-1} < X_N = b,$$

$$c = y_0 < y_1 < \cdots < y_{j-1} < y_j < x_{j+1} < \cdots < y_{M-1} < y_M = d.$$

We write the integral over $\mathcal{R}$ as a repeated integral and then approximate the outer integral by means of the trapezoidal rule, based on the partition of $[a, b]$ by means of the x_k as just described. Using $dx_k \equiv x_k - x_{k-1}$ we obtain

$$\int_{\mathcal{R}} \phi(X)\, dX = \int_a^b \int_c^d \phi(x, y)\, dy\, dx$$

$$= \frac{1}{2} \sum_{k=1}^{N} \left(\int_c^d \phi(x_{k-1}, y)\, dy + \int_c^d \phi(x_k, y)\, dy \right) dx_k.$$

Then we apply the trapezoidal rule again, based on the partition of $[c, d]$ by means of the y_j, to each of the inner integrals. Using $dy_j \equiv y_j - y_{j-1}$ the result is

$$\int_{\mathcal{R}} \phi(X)\, dX = \frac{1}{2} \sum_{k=1}^{N} \left(\frac{1}{2} \sum_{j=1}^{M} (\phi(x_{k-1}, y_{j-1}) + \phi(x_{k-1}, y_j))\, dy_j \right.$$

$$\left. + \frac{1}{2} \sum_{j=1}^{M} (\phi(x_k, y_{j-1}) + \phi(x_k, y_j))\, dy_j \right) dx_k =$$

$$\sum_{k=1}^{N} \sum_{j=1}^{M} \frac{\phi(x_{k-1}, y_{j-1}) + \phi(x_{k-1}, y_j) + \phi(x_k, y_{j-1}) + \phi(x_k, y_j)}{4} dx_k\, dy_j.$$

In other words, the sum, over all of the small rectangles $R_{kj} : x_{k-1} \le x \le x_k, \;\; y_{j-1} \le y \le y_j$, partitioning the larger rectangle $\mathcal{R}$ as a consequence of the two partitionings of $[a, b]$ and $[c, d]$, of the average value of $\phi(x, y)$ over the four corners of R_{kj}, times the area of R_{kj}.

It is not hard to see that this corresponds to the exact integral of the *bilinear* function $\tilde{\phi}(x,y)$ defined on each of the rectangles R_{kj} by

$$\tilde{\phi}(x,y) = \frac{1}{dx_k\,dy_j}[(x-x_{k-1})\,((y-y_{j-1})\,\phi(x_k,y_j)+(y_j-y)\,\phi(x_k,y_{j-1}))$$

$$+\,(x_k-x)\,((y-y_{j-1})\,\phi(x_{k-1},y_j)+(y_j-y)\,\phi(x_{k-1},y_{j-1}))],$$

the graph of which, called a *ruled surface*, can be thought of as a twisted rectangular plane element.

Approximate Integration Based on Triangular Elements Let $\mathcal{R}$ be a region in $\mathbf{R}^2$ and let $\phi(X)$ be a continuous, or piecewise continuous, function defined on $\mathcal{R}$. The method we will describe for approximating $\int_{\mathcal{R}} \phi(X)\,dX$ is valid if $\mathcal{R}$ is bounded by a finite number of simple closed curves. To keep the discussion simple we will suppose $\mathcal{R}$ is bounded by a single curve, $\mathcal{C}$, which is at least piecewise continuously differentiable, meaning that $\mathcal{C}$ consists of a finite number of continuously differentiable segments.

In discussing integrals over $\mathcal{C}$ we have already introduced the notion of a partition on $\mathcal{C}$, consisting of finitely many points X_k. For our present purposes we assume points $X_0, X_1, ..., X_{k-1}, X_k, ..., X_N$, ordered so that they proceed around $\mathcal{C}$ in the mathematically positive (counter-clockwise) direction as k runs from 0 to N, and we will assume $X_N = X_0$; otherwise the X_k are distinct points. A *polygonal approximation* $\mathcal{P}$ to $\mathcal{C}$ can be constructed using the line segments $\mathcal{L}_k$ joining X_{k-1} to X_k, $k = 1, 2, ..., N$; the assumption $X_N = X_0$ ensures that $\mathcal{P}$ is a closed polygon. Then we can introduce additional points X_k, $k = N+1, N+2, ..., N+M$ in the interior of $\mathcal{C}$ and use all of the X_k as vertices, or corners, for a system of non-overlapping triangles, T_j, $j = 1, 2, ..., J$, covering, or very nearly covering, the whole region $\mathcal{R}$. We will refer to such a system as a triangular partitioning of $\mathcal{R}$. There are many ways to construct such partitions, very ambitious (and expensive!) computer codes have been written to construct such partitions for rather complicated regions $\mathcal{R}$.

There is no simple relationship between the total number of points X_k, i.e., $N+M$, and the total number, J, of triangles T_j. However, our procedure is based on a listing of the triangles rather than a listing of the X_k. We select a typical triangle, T, and suppose that its vertices, taken from the X_k, are P, Q and R, listed in positive, or counter-clockwise, order around T; it does not matter where we start. Supposing that

$$P = (p_x, p_y)^*, \;\; Q = (q_x, q_y)^*, \;\; R = (r_x, r_y)^*,$$

we have $\quad \text{area}(T) =$

$$\frac{1}{2}\,\|(Q-P)\times(R-P)\| = \frac{1}{2}\left\|\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ q_x - p_x & q_y - p_y & 0 \\ r_x - p_x & r_y - p_y & 0 \end{pmatrix}\right\|$$

$$= \frac{1}{2}\,(q_x - p_x)\,(r_y - p_y) - \frac{1}{2}\,(q_y - p_y)\,(r_x - p_x)\,.$$

The ordering of P, Q and R allows us to avoid the use of the absolute value in the last expression here; if a clockwise ordering is used, then we must take the absolute value of this expression. Expanding the indicated products we have $\quad \text{area}(T) =$

$$\frac{1}{2}\,(q_x r_y - q_x p_y - p_x r_y + p_x p_y) - \frac{1}{2}\,(q_y r_x - q_y p_x - p_y r_x + p_y p_x)$$

$$= \frac{1}{2}((q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y)) = \frac{1}{2}\det\begin{pmatrix} 1 & 1 & 1 \\ p_x & q_x & r_x \\ p_y & q_y & r_y \end{pmatrix}.$$

This formula shows that the area of a triangle T is very readily computed in terms of its vertices, or “corner points”.

The extension of the trapezoidal method to a triangular partition of $\mathcal{R}$ involves the construction of an approximation to $\int_T \phi(X)\,dX$ for each of the triangles T in the partition, followed by summation to obtain the approximation to $\int_{\mathcal{R}} \phi(X)\,dX$. On T we approximate $\phi(X)$ by $\phi_T(X)$, which is the linear affine function of X whose values coincide with those of $\phi(X)$ at the vertices of T, i.e., using the notation employed above, the values $\phi(P), \phi(Q)$ and $\phi(R)$. Rather than working out $\int_T \phi_T(X)\,dX$ analytically, which is rather tedious, we first consider the function $\phi_{T,P}(X)$ which is the linear affine function such that

$$\phi_{P,T}(P) \;=\; \phi(P),\;\; \phi_{P,T}(Q) \;=\; 0,\;\; \phi_{P,T}(R) \;=\; 0.$$

Thinking of $\phi(P)$ as non-negative, the integral $\int_T \phi_{P,T}(X)\,dX$ coincides with the volume of the three dimensional solid whose four vertices lie at the points P, Q and R in the plane $z = 0$ together with the point whose x and y coordinates are p_x and p_y and whose z coordinate is $\phi(P)$. This solid is a cone whose volume is equal to one third times the area of its base times its height. Thus

$$\int_T \phi_{P,T}(X)\,dX \;=\; \frac{1}{3}\,\phi(P)\,\mathrm{area}(T).$$

Clearly this formula remains valid for the integral even if $\phi(P)$ is negative. Since $\phi_T(X)$ is a linear function of $\phi(P), \phi(Q)$ and $\phi(R)$, we conclude that

$$\begin{aligned}\int_T \phi_T(X)\,dX \;&=\; \frac{1}{3}\,(\phi(P)+\phi(Q)+\phi(R))\;\mathrm{area}(T)\\ &=\; \frac{1}{6}\,(\phi(P)+\phi(Q)+\phi(R))\,\det\begin{pmatrix}1 & 1 & 1\\ p_x & q_x & r_x\\ p_y & q_y & r_y\end{pmatrix}.\end{aligned}$$

Summing over all triangles T in the partition and denoting the vertices of the individual triangles by P_T, Q_T and R_T, we obtain the approximation formula

$$\int_{\mathcal{R}} \phi(X)\,dX \approx$$
$$\sum_{\text{triangles } T} \frac{\phi(P_T)+\phi(Q_T)+\phi(R_T)}{6}\,\det\begin{pmatrix}1 & 1 & 1\\ p_{T,x} & q_{T,x} & r_{T,x}\\ p_{T,y} & q_{T,y} & r_{T,y}\end{pmatrix}.$$

This procedure can be considered an extension of the trapezoidal rule because, just as the familiar one dimensional trapezoidal rule relies on linear approximations to the function $\phi(x)$ on subintervals $[x_{k-1}, x_k]$, agreeing with the original function $\phi(x)$ at the endpoints, this method relies on linear approximations to $\phi(X)$ in triangles T, agreeing with $\phi(X)$ at the vertices of T.

The triangular partition method has, as compared to an attempt to approximate a general region $\mathcal{R}$ by means of rectangles, the advantage of affording a better approximation to the region $\mathcal{R}$. It also has the advantage that it automatically includes a partition of the bounding curve $\mathcal{C}$ (or curves, if there are more of them) which can then be used to compute arclength or line integrals over $\mathcal{C}$ which might be connected with $\int_{\mathcal{R}} \phi(X)\, dX$, as brought out, for example, in Green's Theorem and the Gauss Divergence Theorem.

There is a direct counterpart of this triangular partition scheme which involves partitioning a general region in $\mathbf{R}^3$ by means of *simplices*, each with four vertices. Indeed, there is a counterpart in $\mathbf{R}^n$ for every positive integer n. We will not go into the details of these procedures here.