Integrals on Curves

Integration with Respect to Arc Length Let us suppose we have a curve in $\mathbf{R}^n$:

$$\mathcal{C} : X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \ a \le t \le b,$$

$X(t)$ being continuously differentiable with respect to t. Suppose also that we have a scalar valued function $\rho(X)$ defined and continuous in a region $\mathcal{R} \subset \mathbf{R}^n$ which contains $\mathcal{C}$. When we write

$$\int_{\mathcal{C}} \rho(X)\, ds,$$

what we mean is the integral of $\rho(X)$ over the curve $\mathcal{C}$ **with respect to the arc length** parameter, s, on $\mathcal{C}$. We recall that if the curve $\mathcal{C}$ is parametrized by t as indicated, and if we arbitrarily set the arc length equal to zero at $t = a$, i.e., $s(a) = 0$, then for $t \ge a$ we have

$$s(t) = \int_a^t \left\| X'(u) \right\| du$$

or

$$s'(t) = \left\| X'(t) \right\| = \sqrt{(x_1'(t))^2 + (x_2'(t))^2 + ... + (x_n'(t))^2}.$$

Then we define the **arc length integral**:

$$\int_{\mathcal{C}} \rho(X(s))\, ds = \int_a^b \rho(X(t)) \frac{ds}{dt}\, dt = \int_a^b \rho(X(t)) \left\| X'(t) \right\| dt.$$

Generally the last formula appearing here is the more suitable one for practical computation because the formulae defining the curve $\mathcal{C}$ are ordinarily given in terms of the original parameter, t, rather than in terms of s. It can be seen without great difficulty that the integral with respect to arc length is independent of the particular parametrization of $\mathcal{C}$ as $X(t)$.

Example 1 Suppose a wire, weighing three grams per centimeter of length, is coiled into a helix

$$\mathcal{C}: \begin{array}{rl} x(t) &= 4\cos t \\ y(t) &= 4\sin t\,, \quad 0 \le t \le 4\pi. \\ z(t) &= t \end{array}$$

What is the weight of the wire coil?

Solution What we need here is the arc length integral $\int_{\mathcal{C}} \rho(X)\, ds$, where $\rho(X) \equiv 3$ and s denotes the arc length in centimeters, measured from one end, say the end corresponding to $t = 0$. Since

$$\frac{ds}{dt} = \sqrt{16\sin^2 t + 16\cos^2 t + 1} = \sqrt{17},$$

the required weight is

$$\int_0^{4\pi} 3 \cdot \sqrt{17}\, dt = 12\pi\sqrt{17}.$$

Example 2 Consider another wire, now bent into a planar curve in $\mathbf{R}^2$ described by

$$y = x^2, \ 0 \le x \le 4$$

and tapered so that the density is $\rho = \rho(x) = x$.

Solution Here we will take the coordinate x to be the parameter, so we can write

$$\mathcal{C}: \begin{array}{rl} x &= x \\ y &= x^2 \end{array}.$$

Clearly then

$$s(x) \,=\, \int_0^x \sqrt{1 \,+\, (2\xi)^2}\, d\xi; \;\; \frac{ds}{dx} \,=\, \sqrt{1 \,+\, 4x^2}.$$

Then the weight is

$$\int_0^4 x\sqrt{1+4x^2}\, dx.$$

Taking $r \,=\, x^2$ we have $x \,=\, \frac{1}{2}\frac{dr}{dx}$ and the integral becomes

$$\frac{1}{2}\int_0^{16} \sqrt{1+4r}\, dr \,=\, \frac{1}{12}(1+4r)^{3/2}\Big|_0^{16} \,=\, \frac{1}{12}\Big(65^{3/2} \,-\, 1^{3/2}\Big).$$

Suppose a wire has a density equal to $\rho(X)$ mass units per unit length and is bent into the shape of a curve $\mathcal{C} :\; X \,=\, X(t)$. The **center of gravity** of such a wire is the point $\hat{X}$ such that

$$\int_{\mathcal{C}} \rho(X)(X \,-\, \hat{X})\, ds \,=\, 0 \longrightarrow M\hat{X} \,=\, \int_{\mathcal{C}} X\rho(X)\, ds,$$

where $M \,=\, \int_{\mathcal{C}} \rho(X)\, ds$ is the mass of the wire. In particular, then, in those cases where the density $\rho(X)$ can be written in terms of the arc length parameter s as $\rho(s)$, we have

$$\hat{x} \,=\, \frac{1}{M}\int_{s\,=\,0}^{s\,=\,L} x(s)\rho(s)\, ds, \;\; \hat{y} \,=\, \frac{1}{M}\int_{s\,=\,0}^{s\,=\,L} y(s)\rho(s)\, ds.$$

Example 3 Consider a wire of unit density per unit length bent into a planar semicircle of radius 2. Where is its center of gravity?

Solution It is clear that the mass of the wire is equal to its length, 2π. Taking $x(\theta) \,=\, 2\cos\theta,\; y(\theta) \,=\, 2\sin\theta$, we have

$$\hat{x} \,=\, \frac{2}{2\pi}\int_0^{\pi} \cos\theta \cdot 2\, d\theta \,=\, 0$$

while

$$\hat{y} = \frac{2}{2\pi} \int_0^{\pi} \sin\theta \cdot 2\, d\theta = \frac{2}{\pi} \int_0^{\pi} \sin\theta\, d\theta = \frac{4}{\pi}.$$

The Line Integral Another type of integral defined relative to a curve $\mathcal{C}$ in $\mathbf{R}^n$ is the (not particularly well named) *line integral.* Let us suppose $\mathcal{C}$ is a curve in $\mathbf{R}^n$, lying in a region $\mathcal{R}$ where a vector field $F(X)$ is defined. We want to define what we mean by

$$\int_{\mathcal{C}} F(X) \cdot dX, \text{ or } \int_{\mathcal{C}} F(X)^* dX.$$

Let us suppose the endpoints of $\mathcal{C}$ are A and B. In saying this we imply an *orientation* of the curve $\mathcal{C}$ in the direction from A to B. The same geometric curve, but oriented in the opposite direction from B to A is denoted by $-\mathcal{C}$.

We construct a *partition* of the curve $\mathcal{C}$, consisting of a sequence of points $X_k,\ k = 0, 1, 2, ..., N$ on $\mathcal{C}$ with $X_0 = A,\ X_n = B$ and otherwise ordered in the obvious manner; each X_k lies strictly between X_{k-1} and X_{k+1} on the curve. For $k = 1, 2, ..., N$ we let Ξ_k also be a point on $\mathcal{C}$, lying between (and not necessarily distinct from) X_{k-1} and X_k. We then form the (scalar) sum

$$\sum_{k=1}^{N} F(\Xi_k)^* (X_k - X_{k-1}) =$$

$$F(\Xi_1)^* (X_1 - X_0) + F(\Xi_2)^* (X_2 - X_1) + \cdots + F(\Xi_N)^* (X_N - X_{N-1}).$$

If we assume that $F(X)$ is continuous and $\mathcal{C}$ is *rectifiable*, which means that there is a positive constant M such that for every partition of $\mathcal{C}$ as described above (in particular, for any number N of points X_k) we have

$$\sum_{k=1}^{N} \|X_k - X_{k-1}\| \le M,$$

then one can show the following. Let

$$\delta\,(=\,\delta\,(X_0, X_1, ..., X_N))\;=\;\max_{k=1,2,...,N}\,\{\|X_k - X_{k-1}\|\}\,.$$

Then there is a number $\mathcal{I}$ such that

$$\lim_{\delta\to 0}\,\sum_{k=1}^{N} F(\Xi_k)^*\,(X_k - X_{k-1})\;=\;\mathcal{I}$$

(no matter what collection of partitions we use in letting $\delta\;\to\;0$, which, at the same time, of course, requires that we let $N\;\to\;\infty$). That number is what we define to be the line integral $\int_{\mathcal{C}} F(X)^* dX$. It might be more accurate to call it the *moment* of $F(X)$ over $\mathcal{C}$ but mathematicians are neither less stubborn nor more generally logical than mankind in general; the name line integral is doubtless here to stay.

If (don't confuse n with N or x_k with X_k!)

$$F(X)\;=\;\begin{pmatrix} f_1\,(x_1, x_2, ..., x_n) \\ f_2\,(x_1, x_2, ..., x_n) \\ \vdots \\ f_n\,(x_1, x_2, ..., x_n) \end{pmatrix},$$

the line integral can also be written in the form

$$\sum_{k=1}^{n}\int_{\mathcal{C}} f_k\,(x_1, x_2, ..., x_n)\;dx_k\;=\;\sum_{k=1}^{n}\int_{a_k}^{b_k} f_k\,(x_1, x_2, ..., x_n)\;dx_k,$$

where $A\;=\;(a_1, a_2, ..., a_n)^*\,,\;\;B\;=\;(b_1, b_2, ..., b_n)^*$. The latter integrals are ordinary (scalar) integrals but to compute them requires some preparation. In the k-th integral it is necessary to express each of $x_j,\;\;j = 1, ..., k-1, k+1, ..., n$ as a function of x_k. If it is not possible to express one of these, say x_ℓ, as a single valued function of x_k it may be necessary to re-express the integral as a sum of integrals, each involving a separate branch of the multivalued relationship between x_ℓ and x_k.

Example 1 Let $\mathcal{C}$ be the curve in $\mathbf{R}^2$ which is the graph of $y = x^2,\ 0 \le x \le 2$, and let

$$F(X) = F(x,y) = \begin{pmatrix} 2xy \\ x^2+y^2 \end{pmatrix}.$$

Then

$$\int_{\mathcal{C}} F(X)^* dX = \int_{\mathcal{C}} 2xy\,dx + \int_{\mathcal{C}} \left(x^2+y^2\right)\,dy.$$

On $\mathcal{C}$ the variable x ranges between 0 and 2 while y ranges between 0 and 4. For the range of x given we have $x = \sqrt{y}$. Thus

$$\int_{\mathcal{C}} F(X)^* dX = \int_0^2 2xy\,dx + \int_0^4 \left(x^2+y^2\right) dy$$

$$= \int_0^2 2x^3\,dx + \int_0^4 \left(y+y^2\right) dy = \left.\frac{x^4}{2}\right|_0^2 + \left.\left(\frac{y^2}{2} + \frac{y^3}{3}\right)\right|_0^4 = \frac{112}{3}.$$

Example 2 Same field as in Example 1 and the same relationship between x and y but now over the range $-1 \le x \le 2$. On $-1 \le x < 0$ we have $x = -\sqrt{y}$ while on $0 \le x \le 2$ we have $x = \sqrt{y}$. Thus $\int_{\mathcal{C}} F(X)^* dX =$

$$\left(\int_{-1}^0 2xy\,dx + \int_1^0 \left(x^2+y^2\right) dy\right) + \left(\int_0^2 2xy\,dx + \int_0^4 \left(x^2+y^2\right) dy\right)$$

$$= \left(\int_{-1}^0 2x^3\,dx + \int_1^0 \left(y+y^2\right) dy\right) + \left(\int_0^2 2x^3\,dx + \int_0^4 \left(y+y^2\right) dy\right)$$

$$= \left.\frac{x^4}{2}\right|_{-1}^0 + \left.\left(\frac{y^2}{2} + \frac{y^3}{3}\right)\right|_1^0 + \left.\frac{x^4}{2}\right|_0^2 + \left.\left(\frac{y^2}{2} + \frac{y^3}{3}\right)\right|_0^4 = 36.$$

The line integral has many of the standard properties of ordinary integration. In particular, for two fields $F(X),\ G(X)$ and two scalars $\alpha,\ \beta$ we have the *linearity property*

$$\int_{\mathcal{C}} (\alpha F(x) \, + \, \beta G(X))^* \, dX \; = \; \alpha \int_{\mathcal{C}} F(X)^* dX \, + \, \beta \int_{\mathcal{C}} G(X)^* dX.$$

A further property (note our earlier remarks on orientation) is that

$$\int_{-\mathcal{C}} F(X)^* dX \; = \; - \int_{\mathcal{C}} F(X)^* dX.$$

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves, the first joining A to B, the second joining B to a third point C, then the sum of these curves, $\mathcal{C}_1 + \mathcal{C}_2$, is the curve obtained by following $\mathcal{C}_1$ from A to B and then $\mathcal{C}_2$ from B to C. We then have

$$\int_{\mathcal{C}_1+\mathcal{C}_2} F(X)^* dX \; = \; \int_{\mathcal{C}_1} F(X)^* dX \, + \, \int_{\mathcal{C}_2} F(X)^* dX.$$

If the curve $\mathcal{C}$ is represented parametrically via $X(t), \;\; a \, \leq \, t \, \leq \, b$, the line integral can also be expressed and computed in terms of that parameter, often more easily than the expression in terms of the original variables, x_k, as discussed above. To see this we suppose the points X_k in the partition of the curve $\mathcal{C}$ introduced earlier can be represented as $X_k \, = \, X(t_k), \;\; a \, = \, t_0 \, < \, x_1 \, < \, x_2 \, < \, ... \, < \, x_{N-1} \, < \, x_N \, = \, b$ while the points Ξ_k can be represented as $\Xi_k \, = \, X(\tau_k), \;\; t_{k-1} \, \leq \, \tau_k \, \leq \, t_k, \;\; k \, = \, 1, 2, ..., N$. Then

$$\sum_{k=1}^{N} F(\Xi_k)^* \left(X_k - X_{k-1} \right) \; =$$

$$F(\Xi_1)^* \left(X_1 - X_0 \right) + F(\Xi_2)^* \left(X_2 - X_1 \right) \, + \, \cdots \, + \, F(\Xi_N)^* \left(X_N - X_{N-1} \right).$$

$$= \; F(X(\tau_1))^* \left(X(t_1) - X(t_0) \right) + F(X(\tau_2))^* \left(X(t_2) - X(t_1) \right) +$$

$$\cdots \, + \, F(X(\tau_N))^* \left(X(t_N) - X(t_{N-1}) \right).$$

For each k we have

$$X(t_k) - X(t_{k-1}) \; = \; \int_{t_{k-1}}^{t_k} X'(s) \, ds \; = \; \left(\frac{\int_{t_{k-1}}^{t_k} X'(s) \, ds}{t_k - t_{k-1}} \right) (t_k - t_{k-1}) \, .$$

If we assume $X(t)$ is continuously differentiable, then, as $t_k - t_{k-1} \to 0$,

$$\frac{\int_{t_{k-1}}^{t_k} X^{'}(s)\,ds}{t_k - t_{k-1}} - X^{'}(\tau_k) \to 0$$

and we have

$$\sum_{k=1}^{N} F(\Xi_k)^* (X_k - X_{k-1}) =$$

$$= F(X(\tau_1))^* X^{'}(\tau_1)\,(t_1 - t_0) + F(X(\tau_2))^* X^{'}(\tau_2)\,(t_2 - t_1) +$$

$$\cdots + F(X(\tau_N))^* X^{'}(\tau_N)\,(t_N - t_{N-1}),$$

which is a Riemann sum tending to the integral

$$\int_a^b F(X(t))^* X^{'}(t)\,dt$$

as $\delta t = \max_{k=1,2,\ldots,N} \{t_k - t_{k-1}\} \to 0$. In the end we conclude that

$$\int_{\mathcal{C}} F(X)^* dX = \int_a^b F(X(t))^* X^{'}(t)\,dt.$$

Example 3 If we take $x = t, \; y = t^2$ then the integrals computed in Examples 1 and 2 both take the form

$$\int_{\mathcal{C}} \Big(2xy\,dx + \left(x^2 + y^2\right) dy\Big) = \int_a^b \Big((2t^3) x^{'}(t) + \left(t^2 + t^4\right) y^{'}(t)\Big)\,dt$$

$$= \int_a^b \Big(2t^3 + \left(t^2 + t^4\right) 2t\Big)\,dt = \int_a^b \left(4t^3 + 2t^5\right) dt = \left.\left(t^4 + \frac{t^6}{3}\right)\right|_a^b.$$

Taking $a = 0, \; b = 2$ we obtain the result $\frac{112}{3}$, taking $a = -1, \; b = 2$ we obtain the result 36.

One of the reasons for the importance of the line integral lies in its relationship to the physical/mechanical concept of *work*. Work is often defined, in elementary settings, as the product of force times distance. In a multidimensional setting, if a physical object undergoes a displacement, corresponding to a vector dX and is, at the same time, subject to to a vector force F, the work done on the object by the force F is the inner product:

$$\text{Work} \;=\; F^* \, dX.$$

If the object moves along a smooth curve $\mathcal{C}$ and is subject to a (possibly) varying force $F(x)$, the analogous statement is that the work done on the object by the force field $F(X)$ is given by

$$\text{Work} \;=\; \int_{\mathcal{C}} F(X)^* dX.$$

Many natural force fields are *conservative*; they are formed as the negative gradient of a potential (energy) function $\phi(X)$; $F(X)^* \;=\; \nabla\phi(X)$.

Proposition 1 *If* $F(X)^* \;=\; \nabla\Phi(X),\;\; X \;\in\; \mathcal{R} \subset \mathbf{R}^n$, *for some continuously differentiable potential function* $\phi(X)$ *defined in* $\mathcal{R}$, *and if* $\mathcal{C}$ *is a continuously differentiable (or piecewise continuously differentiable) curve in* $\mathcal{R}$ *with endpoints* A *and* B *(oriented from* A *to* B*), then*

$$\int_{\mathcal{C}} F(X)^* dX \;=\; \int_{\mathcal{C}} \nabla\phi(X) dX \;=\; \phi(B) \;-\; \phi(A).$$

Remark In other words, the line integral of a gradient field over a curve $\mathcal{C}$ depends only on the endpoints of the curve $\mathcal{C}$.

Proof Suppose the curve $\mathcal{C}$ is parametrized as

$$\mathcal{C}: \;\; X \;=\; X(t), \;\; a \;\leq\; t \;\leq\; b, \;\; X(a) \;=\; A, \;\; X(b) \;=\; B,$$

with $X(t)$ a continuously differentiable, or piecewise continuously differentiable (this allows $\mathcal{C}$ to have "corners"), function of t. Then we have

$$\int_{\mathcal{C}} \nabla\phi(X)dX = \int_{\mathcal{C}} \nabla\phi(X(t))\frac{dX}{dt}(t)\,dt = \int_a^b \nabla\phi(X(t))X'(t)\,dt$$

Since the **chain rule** gives $\frac{d}{dt}\phi(X(t)) = \nabla\phi(X(t))X'(t)$, the above yields

$$\int_{\mathcal{C}} \nabla\phi(X)dX = \int_a^b \frac{d}{dt}\phi(X(t))\,dt = \phi(X(b)) - \phi(X(a)) = \phi(B) - \phi(A)$$

and we have the result.

Corollary 1 *Under the same assumptions as set forth in the preceding proposition, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves in $\mathcal{R}$ joining the same two endpoints A and B, both oriented in that direction, then*

$$\int_{\mathcal{C}_1} \nabla\phi(X)\,dX = \int_{\mathcal{C}_2} \nabla\phi(X)\,dX.$$

Proof Both integrals are equal to $\phi(B) - \phi(A)$.

A gradient force field $F(X) = \nabla\phi(X)$ is called a conservative force field because of the following observation. If $\mathcal{C}$ is a *closed curve* in $\mathcal{R}$, joining a point A to A again, then $\int_{\mathcal{C}} \nabla\phi(X)\,dX = 0\ (= \phi(A) - \phi(A))$. This means no work, positive or negative, is done on a object traversing the closed curve $\mathcal{C}$ by the force $F(X) = \nabla\phi(X)^*$; a result which corresponds to the notion of *conservation of energy.*

Implicit in the above proposition and its corollary is the assumption that $\phi(X)$ is a *single-valued* function in $\mathcal{R}$. The following example shows this to be necessary.

Example 4 Let $\phi(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ in the two dimensional region

$$\mathcal{R} = \left\{ (x,y) \in \mathbf{R}^2 \,\middle|\, x^2 + y^2 > 0 \right\}.$$

Let $\mathcal{C}$ be the closed curve in $\mathcal{R}$ consisting of the circle of radius R, centered at the origin. Here we have, in $\mathcal{R}$,

$$\frac{\partial \phi}{\partial x} = \frac{-y}{x^2+y^2}, \quad \frac{\partial \phi}{\partial y} = \frac{x}{x^2+y^2}.$$

Parametrizing the curve $\mathcal{C}$ via $x = x(\theta) = R\cos\theta, y = y(\theta) = R\sin\theta, 0 \leq \theta \leq 2\pi$, we observe that, on $\mathcal{C}$, we have $x^2+y^2 = R^2\cos^2\theta + R^2\sin^2\theta = R^2$ and thus

$$\int_{\mathcal{C}} \nabla\phi(X)\, dX = \int_{\mathcal{C}} \left(\frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy \right)$$

$$= \int_0^{2\pi} \left(\frac{-y(\theta)}{x(\theta)^2 + y(\theta)^2}\, x^{'}(\theta) + \frac{x(\theta)}{x(\theta)^2+y(\theta)^2}\, y^{'}(\theta) \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{-R\sin\theta}{R^2}\,(-R\sin\theta) + \frac{R\cos\theta}{R^2}\,(R\cos\theta) \right) d\theta$$

$$= \int_0^{2\pi} \left(\sin^2\theta + \cos^2\theta \right) d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi,$$

which is certainly not zero. The reason why this does not contradict the results set forth earlier is that $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, being the angular coordinate in the plane, is not single valued; if a point with Cartesian coordinates (x,y) corresponds to an angle θ, then it also corresponds to the angle $\theta + 2k\pi$ for any integer k. As we go around the circle of radius R we start with $\theta = \tan^{-1}\left(\frac{y}{x}\right) = 0$ at the point $(R, 0)$ and end up at that point again, but with $\theta = \tan^{-1}\left(\frac{y}{x}\right) = 2\pi$ at the end.