The Complex Integral, Cauchy’s Theorem and the Cauchy Integral Formula

The Complex Integral and Cauchy’s Theorem  The counterpart to complex differentiation, not surprizingly, is complex integration. This is a complex line integral in much the same spirit as the line integral of the standard vector calculus. But the properties of the complex integral which are critical for developing the complex calculus require that the function $f(z)$ being integrated should be differentiable in the complex case, i.e., $f(z)$ should be analytic.

Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be defined and, at least, continuous in a region $\mathcal{R}$ in the complex plane. Let $\mathcal{C}$ be a rectifiable curve in $\mathcal{R}$, i.e., a curve $C$ having finite arc length. We define

$$\int_{\mathcal{C}} f(z) \, dz = \int_{\mathcal{C}} (u + iv) \, d(x + iy)$$

$$= \int_{\mathcal{C}} (u \, dx - v \, dy) + i \int_{\mathcal{C}} (v \, dx + u \, dy) = \int_{\mathcal{C}} \left( \begin{array}{c} u \\ -v \end{array} \right) \cdot \left( \begin{array}{c} dx \\ dy \end{array} \right) + i \int_{\mathcal{C}} \left( \begin{array}{c} v \\ u \end{array} \right) \cdot \left( \begin{array}{c} dx \\ dy \end{array} \right).$$

This shows $\int_{\mathcal{C}} f(z) \, dz$ to be a complex number whose real and imaginary parts are line integrals over $\mathcal{C}$ of fields $\left( \begin{array}{c} u \\ -v \end{array} \right)$ and $\left( \begin{array}{c} v \\ u \end{array} \right)$. Without further assumptions, however, this integral would be little more than a curiosity.

Now let us suppose that $f(z)$ is analytic in the region $\mathcal{R}$. Then the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

are valid. These are precisely the integrability conditions for the fields $\left( \begin{array}{c} v \\ u \end{array} \right)$ and $\left( \begin{array}{c} u \\ -v \end{array} \right)$, respectively and we conclude that the integral is independent of
path, i.e., if $C_1$ and $C_2$ are two rectifiable curves joining the same endpoints such that the one can be continuously deformed into the other without leaving the region $R$, then $\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz$. An immediate restatement of this is what is called Cauchy’s Theorem.

**Theorem 1** If $C$ is a closed, rectifiable curve which, together with its interior, lies in a region $R$ where $f(z)$ is analytic, then $\int_C f(z) \, dz = 0$.

The proof is immediate, applying Green’s Theorem to the real and imaginary parts of the integral and using the Cauchy–Riemann conditions as noted above.

It is crucial that $f(z)$ should be analytic throughout the interior of a closed curve $C$ in order for the above result to be valid. The most significant example is the following.

**Example** Let $f(z) = \frac{1}{z - z_0}$ for some fixed $z_0$ and let $C_r$ be the circle of radius $r > 0$ centered at $z_0$, oriented in the positive, counterclockwise, direction. In this case $f(z)$ has a singularity at $z_0$ and is not analytic there. Using the polar form, $z - z_0 = re^{i\theta}$ on $C_r$ while $dz = re^{i\theta} \, d\theta$ there. We have

$$\int_{C_r} \frac{1}{z - z_0} \, dz = \int_0^{2\pi} \frac{1}{re^{i\theta}} re^{i\theta} \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi i.$$ 

**The Complex Antiderivative (Indefinite Integral)**

Carrying this further, if the region $R$ is simply connected, the integrable fields $\left( \frac{u}{-v} \right)$ and $\left( \frac{v}{u} \right)$ have single valued potential functions $U(x, y)$ and $V(x, y)$, as developed in the section on integrable fields and potentials.
We have
\[ \frac{\partial U}{\partial x} = u, \quad \frac{\partial U}{\partial y} = -v, \quad \frac{\partial V}{\partial x} = v, \quad \frac{\partial V}{\partial y} = u. \]

If we then form the complex function
\[ F(z) = F(x + iy) = U(x, y) + iV(x, y) \]
we see that if \( C \) is a curve joining \( z_0 \) to \( z \) we have
\[ F(z) = U(x, y) + iV(x, y) = U(x_0, y_0) + iV(x_0, y_0) + \int_C (\frac{u}{d\xi} - v) \cdot (\frac{d\xi}{d\eta}) + i (\frac{v}{d\eta} + u) \cdot (\frac{d\xi}{d\eta}) \]
\[ = U(x_0, y_0) + iV(x_0, y_0) + \int_C f(\zeta) d\zeta = F(z_0) + \int_C f(\zeta) d\zeta. \]
The conditions \( \frac{\partial U}{\partial x} = u, \frac{\partial U}{\partial y} = -v, \frac{\partial V}{\partial x} = v, \frac{\partial V}{\partial y} = u \), show that \( U(x, y) \) and \( V(x, y) \) also satisfy the Cauchy–Riemann conditions, so \( F(z) \) is differentiable and
\[ F'(z) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial x} = u(x, y) + i v(x, y) = f(z). \]
Thus \( F(z) \), as defined by the indicated complex integral, is an antiderivative of \( f(z) \). This result can be regarded as the complex fundamental theorem of calculus.

**The Cauchy Integral Formula**  As a result of Cauchy’s Theorem and its extension to regions which are not simply connected, we can now obtain a formula, called the *Cauchy Integral Formula* which expresses the value of an analytic function \( f(z) \) inside a simple closed curve, \( C \), in terms of its values on that curve. It then turns out that the formula allows us to obtain very strong properties of \( f(z) \) which would be difficult to deduce in any other way. First of all, the statement of the theorem.
Theorem 2 Let \( f(z) \) be analytic in a region \( \mathcal{R} \) of the complex plane which includes a simple closed curve \( \mathcal{C} \) and its interior. Let \( z_0 \) be a point in the interior of \( \mathcal{C} \). Then, with \( \mathcal{C} \) oriented in the positive direction, we have

\[
\int_{\mathcal{C}} \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0).
\]

Proof Since \( z_0 \) lies in the interior of \( \mathcal{C} \) we can find \( r > 0 \) such that \( \mathcal{C}_r = \{ z \mid |z - z_0| < r \} \), together with its interior, is contained in the interior of \( \mathcal{C} \). Since \( f(z) \) is differentiable at \( z_0 \) we can write, for \( z \) near \( z_0 \),

\[
f(z) = f(z_0) + f'(z_0)(z - z_0) + e(z, z_0),
\]

where

\[
\lim_{z \to z_0} \frac{e(z, z_0)}{|z - z_0|} = 0.
\]

Since \( f(z)/(z - z_0) \) is analytic in the region between \( \mathcal{C} \) and \( \mathcal{C}_r \), we have

\[
\int_{\mathcal{C}} \frac{f(z)}{z - z_0} \, dz = \int_{\mathcal{C}_r} \frac{f(z)}{z - z_0} \, dz
\]

\[
= \int_{\mathcal{C}_r} \frac{f(z_0)}{z - z_0} \, dz + \int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz.
\]

Since \( f(z_0) \) is constant, by our previous result we have

\[
\int_{\mathcal{C}_r} \frac{f(z_0)}{z - z_0} \, dz = f(z_0) \int_{\mathcal{C}_r} \frac{1}{z - z_0} \, dz = 2\pi i f(z_0).
\]

Letting \( r \to 0 \), we will clearly have

\[
\int_{\mathcal{C}} \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0) + \lim_{r \to 0} \int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz.
\]

Thus it remains only to get rid of the term \( \int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz \). Using the expression obtained above this integral becomes

\[
\int_{\mathcal{C}_r} \frac{f'(z_0)(z - z_0) + e(z, z_0)}{z - z_0} \, dz = \int_{\mathcal{C}_r} 1 \, dz + \int_{\mathcal{C}_r} \frac{e(z, z_0)}{z - z_0} \, dz.
\]
The first term is zero by the Cauchy Integral Theorem because the constant function 1 is analytic on $C_r$ and in its interior. On the other hand we have

$$\left| \int_{C_r} \frac{e(z, z_0)}{z - z_0} \, dz \right| < \mathcal{L}(C_r) \max_{C_r} \left\{ \frac{|e(z, z_0)|}{|z - z_0|} \right\},$$

where $\mathcal{L}(C_r) = 2\pi r$, the length of $C_r$. Since $|z - z_0| = r$, we have

$$\left| \int_{C_r} \frac{e(z, z_0)}{z - z_0} \, dz \right| < 2\pi \max_{C_r} \left\{ |e(z, z_0)| \right\}$$

which clearly tends to 0 as $r \to 0$. Then we have

$$\int_{C} \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0)$$

and the theorem is proved.

This theorem has a huge number of important consequences. Replacing $z$ by $w$ and $z_0$ by $z$, we have, for $z \in \text{Int } C$, the formula

$$f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(w)}{w - z} \, dw,$$

which shows, in particular, that the values of $f(z)$ throughout the interior of $C$ are determined by the values $f(w)$ with $w$ on $C$.

**Functions Constructed via the Cauchy Formula** The Cauchy Integral Formula, as applied to an analytic function above, prompts another question. Let $\mathcal{R}$ and $\mathcal{C}$ be as in the theorem above and let $f(w)$ be an arbitrary continuous complex function, or just an integrable function, defined for $w \in \mathcal{C}$. Then we can define a function $\hat{f}(z)$ for $z$ in the interior of $\mathcal{C}$ by

$$\hat{f}(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w)}{w - z} \, dw.$$
We can then ask: what are the properties of \( \hat{f}(z) \)? Is it analytic as a functions of \( z \)? How is it related to \( f(w) \)? The answers to these questions are not entirely straightforward. Yes, it is true that \( \hat{f}(z) \) exists and is analytic in the interior of \( \mathcal{C} \) but it may or may not have any natural relationship to \( f(w) \) as defined for \( w \in \mathcal{C} \). Suppose, e.g., that \( \mathcal{C}_r \) is the circle of radius \( r \), centered at the origin, and let \( f(w) = w^{-1} \) on \( \mathcal{C}_r \). Then for \( z \) in the interior of \( \mathcal{C}_r \)

\[
\hat{f}(z) = \frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{1}{w-z} \, dw = \frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{1}{w(w-z)} \, dw = \frac{1}{2\pi i z} \int_{\mathcal{C}_r} \left( \frac{1}{w-z} - \frac{1}{w} \right) \, dw.
\]

We have seen earlier that

\[
\int_{\mathcal{C}_r} \frac{1}{w} \, dw = 2\pi i.
\]

Let \( |z| = \rho < r \) and let \( \mathcal{C}_{z,r-\rho} \) be the circle of radius \( r - \rho > 0 \) centered at \( z \). Since \( (w-z)^{-1} \) is analytic in the region outside \( \mathcal{C}_{z,r-\rho} \) and inside \( \mathcal{C}_r \) the integral over the boundary of that region, which is the integral over \( \mathcal{C}_r - \mathcal{C}_{z,r-\rho} \), is zero. Then

\[
\int_{\mathcal{C}_r} \frac{1}{w-z} \, dw = \int_{\mathcal{C}_{z,r-\rho}} \frac{1}{w-z} \, dw = \int_{\mathcal{C}_{z,r-\rho}} \frac{1}{w-z} \, d(w-z) = 2\pi i
\]

by the same argument as gave us that result for the integral of \( 1/w \) over \( \mathcal{C}_r \). Putting these results together it follows that \( \hat{f}(z) \equiv 0 \) for \( z \) in the interior of \( \mathcal{C}_r \), a value not related in any obvious way to the function \( f(w) = 1/w \). If, on the other hand we were to take \( f(w) = w \) on \( \mathcal{C}_r \) then, \( f(w) \) is analytic on \( \mathcal{C}_r \) and its interior and the Cauchy formula would give us \( \hat{f}(z) \equiv z \). We will obtain results later explaining this seemingly strange behavior.

**Uniformly Convergent Sequences of Analytic Functions** The ambiguity noted above is not present when the function \( f(z) \) is the uniform limit of a sequence of analytic functions. We suppose \( \mathcal{R} \) is a region (open,
connected set) in the complex plane and \( \{ f_k(z) \} \) is a sequence of analytic functions defined on \( \mathcal{R} \) converging uniformly on \( \mathcal{R} \) to a function \( f(z) \). Then we claim \( f(z) \) is analytic on \( \mathcal{R} \).

To prove this we let \( z \in \mathcal{R} \) and let \( C \) be a simple closed curve in \( \mathcal{R} \) which contains \( z \) in its interior. Since the \( f_k(z) \) are assumed analytic in \( \mathcal{R} \), for each \( k \) we have
\[
f_k(z) = \frac{1}{2\pi i} \int_C \frac{f_k(w)}{w-z} \, dw.
\]
From the assumed uniform convergence to \( f(z) \),
\[
f(z) = \lim_{k \to \infty} f_k(z)
\]
and
\[
\lim_{k \to \infty} \int_C \frac{f_k(w)}{w-z} \, dw = \int_C \frac{f(w)}{w-z} \, dw
\]
from which it follows that
\[
f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} \, dw.
\]
The right hand side of this identity is differentiable with respect to \( z \); indeed
\[
\frac{d}{dz} \left( \int_C \frac{f(w)}{w-z} \, dw \right) = \int_C \frac{f(w)}{(w-z)^2} \, dw.
\]
Accordingly, the left hand side is also differentiable at \( z \) with
\[
f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} \, dw.
\]
Since \( z \) is an arbitrary point in \( \mathcal{R} \) we conclude that \( f(z) \) is analytic in \( \mathcal{R} \) and we have the desired result.