

Stokes Theorem and its Applications

Theorem 1 *Let $F(X) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$ be a continuously differentiable vector field in a domain $\mathcal{D} \subset \mathbb{R}^3$ containing a simple closed curve \mathcal{C} and an orientable surface Σ such that \mathcal{C} is the boundary of Σ . For points $(x, y, z) \in \Sigma$, let $\nu(x, y, z)$ be the unit normal to Σ agreeing, via the right hand rule, with the selected orientation of the curve \mathcal{C} . Then*

$$\int_{\Sigma} \nabla \times F(X) \cdot \nu \, d\sigma = \int_{\mathcal{C}} F(X) \cdot dX.$$

Important Applications Stokes Theorem, sometimes combined with Gauss' Theorem, provides a powerful tool for working with various sorts of surface and line integrals. It has important physical applications.

Proof We first give the proof in a special case. We assume the surface Σ is “projectable” onto each of the x, y, y, z and z, x coordinate planes. This means that the equation defining Σ , which we may assume to take the form $\phi(x, y, z) = 0$, is solvable for each of the variables x, y and z ; thus we have

$$z = \varphi(x, y), \quad x = \psi(y, z), \quad y = \theta(x, z).$$

We let $\mathcal{C}_{x,y}$, $\mathcal{C}_{y,z}$ and $\mathcal{C}_{z,x}$ be the two dimensional plane curves which are the projections of \mathcal{C} onto the x, y, y, z and z, x planes, respectively, and we let $\Sigma_{x,y}$, $\Sigma_{y,z}$ and $\Sigma_{z,x}$ be the regions interior to the plane curves $\mathcal{C}_{x,y}$, $\mathcal{C}_{y,z}$ and $\mathcal{C}_{z,x}$, respectively, in those planes.

We note that $F(X)$ can be decomposed as

$$\begin{aligned} F(X) &= \begin{pmatrix} f(x, y, z) \\ g(x, y, z) \\ h(x, y, z) \end{pmatrix} = \frac{1}{2} \left[\begin{pmatrix} f \\ g \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ g \\ h \end{pmatrix} + \begin{pmatrix} f \\ 0 \\ h \end{pmatrix} \right] \\ &= F_{x,y}(X) + F_{y,z}(X) + F_{z,x}(X). \end{aligned}$$

Starting with the field $F_{x,y}$ and $z = \varphi(x, y)$, we have

$$\begin{aligned} F_{x,y}(x, y, z) &= F_{x,y}(x, y, \varphi(x, y)) \\ &= f(x, y, \varphi(x, y)) \mathbf{i} + g(x, y, \varphi(x, y)) \mathbf{j} \equiv \tilde{f}(x, y) \mathbf{i} + \tilde{g}(x, y) \mathbf{j}. \end{aligned}$$

Because this field depends only on x and y we can treat it as if it were defined in a region of the x, y plane containing the projected curve $\mathcal{C}_{x,y}$ and its interior $\Sigma_{x,y}$. We compute, using the chain rule,

$$\begin{aligned} \frac{\partial \tilde{g}}{\partial x} - \frac{\partial \tilde{f}}{\partial y} &= \frac{\partial}{\partial x} \tilde{g}(x, y, \varphi(x, y)) - \frac{\partial}{\partial y} \tilde{f}(x, y, \varphi(x, y)) \\ &= \frac{\partial g}{\partial x}(x, y, \varphi(x, y)) + \frac{\partial g}{\partial z}(x, y, \varphi(x, y)) \frac{\partial \varphi}{\partial x}(x, y) \\ &\quad - \frac{\partial f}{\partial y}(x, y, \varphi(x, y)) - \frac{\partial f}{\partial z}(x, y, \varphi(x, y)) \frac{\partial \varphi}{\partial y}(x, y) \\ &= \left(-\frac{\partial g}{\partial z} \mathbf{i} + \frac{\partial f}{\partial z} \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \right) (x, y, \varphi(x, y)) \cdot \left(-\frac{\partial \varphi}{\partial x} \mathbf{i} - \frac{\partial \varphi}{\partial y} \mathbf{j} + \mathbf{k} \right) (x, y). \end{aligned}$$

Applying Green's Theorem we have

$$\begin{aligned} \int_{\mathcal{C}_{x,y}} (\tilde{f}(x, y) dx + \tilde{g}(x, y) dy) &= \int_{\Sigma_{x,y}} \left(\frac{\partial \tilde{g}}{\partial x} - \frac{\partial \tilde{f}}{\partial y} \right) dx dy \\ &= \int_{\Sigma} \left(-\frac{\partial g}{\partial z} \mathbf{i} + \frac{\partial f}{\partial z} \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \right) \cdot \frac{\left(-\frac{\partial \varphi}{\partial x} \mathbf{i} - \frac{\partial \varphi}{\partial y} \mathbf{j} + \mathbf{k} \right)}{\sqrt{1 + \left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2}} d\sigma \\ &= \int_{\Sigma} \left(-\frac{\partial g}{\partial z} \mathbf{i} + \frac{\partial f}{\partial z} \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \right) (x, y, \varphi(x, y)) \cdot \nu(x, y, \varphi(x, y)) d\sigma. \end{aligned}$$

Since dx and dy at a point $(x, y) \in \mathcal{C}_{x,y}$ agree with dx and dy at the corresponding point $(x, y, \varphi(x, y)) \in \mathcal{C}$, we have

$$\int_{\mathcal{C}} (f(x, y, z) dx + g(x, y, z) dy) = \int_{\Sigma} \left(-\frac{\partial g}{\partial z} \mathbf{i} + \frac{\partial f}{\partial z} \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \right) \cdot \nu d\sigma.$$

Using $x = \psi(y, z)$, $\mathcal{C}_{y,z}$ and $\Sigma_{y,z}$, essentially the same argument yields

$$\int_{\mathcal{C}} (g(x, y, z) dy + h(x, y, z) dz) = \int_{\Sigma} \left(\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} - \frac{\partial h}{\partial x} \mathbf{j} + \frac{\partial g}{\partial x} \mathbf{k} \right) \cdot \nu d\sigma.$$

Finally, using $y = \theta(x, z)$, $\mathcal{C}_{z,x}$ and $\Sigma_{z,x}$ we have

$$\int_{\mathcal{C}} (f(x, y, z) dy + h(x, y, z) dz) = \int_{\Sigma} \left(\frac{\partial h}{\partial y} \mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} - \frac{\partial f}{\partial y} \mathbf{k} \right) \cdot \nu d\sigma.$$

Adding these three equations and dividing by 2 we obtain

$$\begin{aligned} \int_{\mathcal{C}} F(X)^* dX &= \int_{\mathcal{C}} F(X) \cdot dX = \int_{\mathcal{C}} (f(x, y, z) dx + g(x, y, z) dy + h(x, y, z) dz) \\ &= \int_{\Sigma} \left(\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \right) \cdot \nu d\sigma \\ &= \int_{\Sigma} \mathbf{curl} F(X) \cdot \nu(X) d\Sigma \end{aligned}$$

and the proof for special surfaces Σ , as described, is complete.

An arbitrary orientable surface Σ (i.e., a surface having two distinct sides, as opposed to a Moebius strip, e.g.) can be subdivided into a finite number of smaller surfaces Σ_k , $k = 1, 2, \dots, K$, satisfying the hypotheses of the special surface considered above, and bounded by curves \mathcal{C}_k , such that the Σ_k are disjoint except for intersections at their bounding curves, $\Sigma = \cup_{k=1}^K \Sigma_k$, and $\sum_{k=1}^K \mathcal{C}_k = \mathcal{C}$, by which we mean that the curves \mathcal{C}_k are oriented in the positive direction relative to $\nu(X)$ and the right hand rule, and all subarcs of the \mathcal{C}_k which are not subarcs of \mathcal{C} are covered twice, in opposite directions, so that the corresponding line integrals cancel. Applying our result to each of these surfaces and their bounding curves we complete the proof of the theorem by observing that

$$\begin{aligned} \int_{\Sigma} \mathbf{curl} F(X)^* \nu(X) dX &= \sum_{k=1}^K \int_{\Sigma_k} \mathbf{curl} F(X)^* \nu(X) dX \\ &= \sum_{k=1}^K \int_{\mathcal{C}_k} F(X)^* dX = \int_{\mathcal{C}} F(X)^* dX. \end{aligned}$$

Example 1 We consider the field in R^3 :

$$F(X) = \frac{1}{2} (z^2 \mathbf{i} + x^2 \mathbf{j} + y^2 \mathbf{k}),$$

whose curl is readily seen to be

$$\nabla \times F(X) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}.$$

We let Σ denote the paraboloid surface

$$\Sigma = \{(x, y, z) \mid 4z + x^2 + y^2 - 1 = 0 \text{ or } z = 1 - (x^2 + y^2)/4\}.$$

The unit normal to this surface with positive component in the z direction is

$$\nu(X) = (x \mathbf{i} + y \mathbf{j} + 2 \mathbf{k}) / \sqrt{4 + x^2 + y^2}.$$

The surface integral in Stokes theorem is thus

$$\begin{aligned} & \iint_{\Sigma} (y \mathbf{i} + z \mathbf{j} + x \mathbf{k}) \cdot \left((x \mathbf{i} + y \mathbf{j} + 2 \mathbf{k}) / \sqrt{4 + x^2 + y^2} \right) \sqrt{1 + \frac{x^2}{4} + \frac{y^2}{4}} \, dx dy \\ &= 2 \iint_{\Sigma} (y \mathbf{i} + z \mathbf{j} + x \mathbf{k}) \cdot (x \mathbf{i} + y \mathbf{j} + 2 \mathbf{k}) \, dx dy = 2 \iint_{\mathcal{D}} (xy + zy + 2x) \, dx dy, \end{aligned}$$

where \mathcal{D} is the disk of radius 2 centered at the origin in the x, y plane. It is tedious but not too difficult to see that this integral is zero. Applying Stokes theorem, the integral must agree with

$$\frac{1}{2} \int_{\mathcal{C}} (z^2 dx + x^2 dy + y^2 dz) = \frac{1}{2} \int_{\mathcal{C}} x^2 dy$$

because $z \equiv 0$ on \mathcal{C} in the x, y plane. Changing to polar coordinates this last integral is computed somewhat more easily as

$$\begin{aligned} & \frac{1}{2} \int_0^{2\pi} 4 \cos^2 \theta d(2 \sin \theta) = 4 \int_0^{2\pi} \cos^3 \theta d\theta \\ &= 4 \int_0^{2\pi} (1 - \sin^2 \theta) \cos \theta d\theta = \left(\sin \theta - \frac{\sin^3 \theta}{3} \right) \Big|_0^{2\pi} = 0. \end{aligned}$$

Equivalent Surfaces We call two surfaces Σ_1 and Σ_2 , **equivalent** if they are bounded by a common simple closed curve \mathcal{C} , or if the boundaries of each consist of the same finite set of such curves, oriented in the same way relative to the corresponding normals $\nu_1(X)$ and $\nu_2(X)$ on Σ_1 and Σ_2 , respectively, following the right hand rule. (Thus if we take Σ_1 and Σ_2 to be the oppositely oriented versions of the same surface, Σ , they are not equivalent because the common

Example 2 The surfaces

$$z = \sqrt{1 - x^2 - y^2} \quad \text{and} \quad z = -\sqrt{1 - x^2 - y^2},$$

both oriented so that the upward side of the surface (positive z direction) is considered to be the positive side, are equivalent; the common bounding curve is the unit circle $x^2 + y^2 = 1$ in the plane $z = 0$.

If Σ_1 and Σ_2 are equivalent surfaces as just defined, and if $F(X)$ is a continuously differentiable field defined in a region including these surfaces and the three dimensional region \mathcal{R} lying between these two surfaces, then, applying Stokes Theorem twice,

$$\int_{\Sigma_1} \nabla \times F(X) \cdot \nu_1 d\sigma = \int_{\mathcal{C}} F(X) \cdot dX = \int_{\Sigma_2} \nabla \times F(X) \cdot \nu_2 d\sigma,$$

where ν_1, ν_2 are the positively directed unit normals to Σ_1, Σ_2 , respectively, in each case relative to the chosen orientation of \mathcal{C} .

Another way in which we can see this is to let $\Sigma = \Sigma_1 \cup \Sigma_2$; then Σ is the bounding surface for \mathcal{R} and the unit outward normal ν to Σ agrees with ν_1 on Σ_1 and with $-\nu_2$ on Σ_2 . Since one of the identities we have seen earlier is $\nabla \cdot \nabla \times F(X) \equiv 0$, application of Gauss's Theorem gives

$$\int_{\Sigma} \nabla \times F(X) \cdot \nu d\sigma = \int_{\mathcal{R}} \nabla \cdot \nabla \times F(X) dX = 0.$$

But since

$$\int_{\Sigma} \nabla \times F(X) \cdot \nu d\sigma = \int_{\Sigma_1} \nabla \times F(X) \cdot \nu d\sigma + \int_{\Sigma_2} \nabla \times F(X) \cdot \nu d\sigma$$

$$= \int_{\Sigma_1} \nabla \times F(X) \cdot \nu_1 d\sigma - \int_{\Sigma_2} \nabla \times F(X) \cdot \nu_2 d\sigma,$$

the last two integrals must be equal, as before.

Example 3 Let Σ be the hemisphere in R^3 :

$$z = \sqrt{1 - (x - 1)^2 - (y - 1)^2}, \text{ or } z - \sqrt{1 - (x - 1)^2 - (y - 1)^2} = 0,$$

and let

$$F(X) = F(x, y, z) = yz \mathbf{i} - xz \mathbf{j} + yx \mathbf{k}.$$

We want to compute $\int_{\Sigma} \nabla \times F(X) \cdot \nu d\sigma$. If we follow the direct approach

$$\nabla \times F(x, y, z) = 2x \mathbf{i} - 2z \mathbf{k},$$

$$\nu(x, y, z) = \frac{-\frac{(x-1)}{s} \mathbf{i} - \frac{(y-1)}{s} \mathbf{j} + \mathbf{k}}{\sqrt{1 + \frac{(x-1)^2}{s^2} + \frac{(y-1)^2}{s^2}}} = -(x-1) \mathbf{i} - (y-1) \mathbf{j} + s \mathbf{k},$$

where $s = \sqrt{1 - (x - 1)^2 - (y - 1)^2}$ and $d\sigma =$

$$\begin{aligned} & \sqrt{1 + \left(\frac{-(x-1)}{\sqrt{1 - (x-1)^2 - (y-1)^2}} \right)^2 + \left(\frac{-(y-1)}{\sqrt{1 - (x-1)^2 - (y-1)^2}} \right)^2} dx dy \\ &= \frac{1}{\sqrt{1 - (x-1)^2 - (y-1)^2}} dx dy = \frac{1}{s} dx dy. \end{aligned}$$

Thus the integral becomes

$$\int_{(x-1)^2 + (y-1)^2 \leq 1} \begin{pmatrix} 2x \\ 0 \\ -2z \end{pmatrix} \cdot \begin{pmatrix} -(x-1) \\ -(y-1) \\ s z \end{pmatrix} \frac{1}{s} dx dy$$

which is not easy to calculate.

On the other hand, letting \mathcal{C} be the circle of radius 1 about the point $x = 1, y = 1$ in the x, y plane, where $z = 0$ and $dz = 0$, Stokes Theorem gives the value of the integral as (\cdot indicates ordinary multiplication here)

$$\int_{\mathcal{C}} F(X)^* dX = \int_{\mathcal{C}} (y \cdot 0 dx - x \cdot 0 dy + yx dz) = 0.$$