The Gauss Divergence Theorem

Theorem 1 *Let $F(X)$ be a continuously differentiable vector field in a domain $\mathcal{D} \subset R^n$. Let $\mathcal{R} \subset \mathcal{D}$ be a closed, bounded region whose boundary is a smooth surface, $\Sigma \subset \mathcal{D}$. For each point $x \in \Sigma$ let $\nu(X)$ be the unit outward, or "exterior", normal to Σ with respect to the region $\mathcal{R}$. Then, with $dX \equiv dx_1 dx_2 ... dx_n$ and with $d\sigma$ indicating integration with respect to surface area on Σ,*

$$\int_{\mathcal{R}} \nabla \cdot F(X)\, dX \;=\; \int_{\Sigma} F(X) \cdot \nu(X)\, d\sigma.$$

Proof We will give the proof for the case $n = 3$; much the same proof can be given in the general n-dimensional situation. Supposing that

$$F(X) \;=\; F(x,y,z) \;=\; \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{pmatrix},$$

we have

$$\nabla \cdot F(X) \;=\; \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right)(x,y,z).$$

Thus

$$\int_{\mathcal{R}} \nabla \cdot F(X)\, dX \;=\; \int_{\mathcal{R}} \frac{\partial f}{\partial x}\, dX \;+\; \int_{\mathcal{R}} \frac{\partial g}{\partial y}\, dX \;+\; \int_{\mathcal{R}} \frac{\partial h}{\partial z}\, dX.$$

Initially we will concentrate our attention on the last integral appearing here.

We will also initially suppose (see figure on page 5) that the bounding surface Σ for $\mathcal{R}$ consists of an "upper" surface and a "lower" surface:

$$\Sigma_\tau : \; \phi(x,y,z) \;=\; z - \tau(x,y) \;=\; 0; \; \Sigma_\rho : \; \psi(x,y,z) \;=\; z - \rho(x,y) \;=\; 0,$$

together with some possible surface components parallel to the z-axis which we will denote collectively by Σ_z.

If we take a point on the upper surface, namely $(x, y, \tau(x, y))$, the unit normal vector there is

$$\nu_\tau(X) = \frac{-\mathbf{i}\frac{\partial\tau}{\partial x} - \mathbf{j}\frac{\partial\tau}{\partial y} + \mathbf{k}}{\sqrt{\left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2 + 1}}.$$

On the lower surface the unit outward normal vector is

$$\nu_\rho(X) = \frac{\mathbf{i}\frac{\partial\rho}{\partial x} + \mathbf{j}\frac{\partial\rho}{\partial y} - \mathbf{k}}{\sqrt{\left(\frac{\partial\rho}{\partial x}\right)^2 + \left(\frac{\partial\rho}{\partial y}\right)^2 + 1}}.$$

On the upper surface, $z = \tau(x, y)$, the element of surface area is

$$d\sigma = \sqrt{\left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2 + 1}\; dx\, dy,$$

while on the lower surface the corresponding surface area element is

$$d\sigma = \sqrt{\left(\frac{\partial\rho}{\partial x}\right)^2 + \left(\frac{\partial\rho}{\partial y}\right)^2 + 1}\; dx\, dy.$$

Therefore, with $\mathcal{R}_{xy}$ denoting the projection of $\mathcal{R}$ onto the x, y plane,

$$\begin{aligned}\int_{\mathcal{R}} \frac{\partial h}{\partial z}(x, y, z)\, dx\, dy\, dz &= \int_{\mathcal{R}_{xy}} \left(\int_{z=\rho(x,y)}^{z=\tau(x,y)} \frac{\partial h}{\partial z}(x, y, z)\, dz \right) dx\, dy \\ &= \int_{\mathcal{R}_{xy}} \left(h(x, y, \tau(x, y)) - h(x, y, \rho(x, y)) \right) dx\, dy \\ &= \int_{\Sigma_\tau} h(x, y, \tau(x, y)) \frac{d\sigma}{\sqrt{\left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2 + 1}} \\ &\quad - \int_{\Sigma_\rho} h(x, y, \rho(x, y)) \frac{d\sigma}{\sqrt{\left(\frac{\partial\rho}{\partial x}\right)^2 + \left(\frac{\partial\rho}{\partial y}\right)^2 + 1}}.\end{aligned}$$

But we clearly have

$$\frac{1}{\sqrt{\left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2 + 1}} = \mathbf{k}\cdot\nu_\tau(X), \quad \frac{-1}{\sqrt{\left(\frac{\partial\rho}{\partial x}\right)^2 + \left(\frac{\partial\rho}{\partial y}\right)^2 + 1}} = \mathbf{k}\cdot\nu_\rho(X).$$

Consequently

$$\int_{\mathcal{R}} \frac{\partial h}{\partial z}(x,y,z)\,dx\,dy\,dz$$
$$= \int_{\Sigma_\tau} h(x,y,z)\,\mathbf{k}\cdot\nu_\tau(X)\,d\sigma + \int_{\Sigma_\rho} h(x,y,z)\,\mathbf{k}\cdot\nu_\rho(X)\,d\sigma.$$

On Σ_z we have $\mathbf{k}\cdot\nu(X) \equiv 0$, so the above can be extended to

$$\int_{\mathcal{R}} \frac{\partial h}{\partial z}(x,y,z)\,dx\,dy\,dz = \int_{\Sigma} h(x,y,z)\,\mathbf{k}\cdot\nu(X)\,d\sigma.$$

Complicated regions $\mathcal{R}$ can be reduced, by means of cuts parallel to the z-axis to regions satisfying the hypothesis above. So we may assume the result obtained so far is not limited with respect to the geometry of $\mathcal{R}$. (See figure on p. 5.)

In the same way we can see that

$$\int_{\mathcal{R}} \frac{\partial f}{\partial x}(x,y,z)\,dx\,dy\,dz = \int_{\Sigma} f(x,y,z)\,\mathbf{i}\cdot\nu(X)\,d\sigma;$$

$$\int_{\mathcal{R}} \frac{\partial g}{\partial y}(x,y,z)\,dx\,dy\,dz = \int_{\Sigma} g(x,y,z)\,\mathbf{j}\cdot\nu(X)\,d\sigma.$$

Adding the three results separately obtained for f, g and h we have

$$\int_{\mathcal{R}} \nabla\cdot F(X)\,dX = \int_{\mathcal{R}} \left(\frac{\partial f}{\partial x}(x,y,z) + \frac{\partial g}{\partial y}(x,y,z) + \frac{\partial h}{\partial z}(x,y,z)\right) dx\,dy\,dz$$
$$= \int_{\Sigma} (f(x,y,z)\,\mathbf{i} + g(x,y,z)\,\mathbf{j} + h(x,y,z)\,\mathbf{k})\cdot\nu(X)\,d\sigma$$
$$= \int_{\Sigma} F(X)\cdot\nu(X)\,d\sigma$$

and the theorem is proved.

Example 1 Let $F(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and let Σ be the sphere of radius 1 in R^3. We first compute $= \int_\Sigma F(X) \cdot \nu(X)\, d\sigma$. Here we have

$$\nu(X) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow F(X) \cdot \nu(X) = \frac{x^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \sqrt{x^2 + y^2 + z^2} \equiv 1 \text{ on } \Sigma.$$

So we have

$$\int_\Sigma F(X) \cdot \nu(X)\, d\sigma = 1 \cdot \int_\Sigma d\sigma = 4\pi.$$

On the other hand, with $\mathcal{R}$ denoting the unit ball inside the unit sphere Σ as described above, we have

$$\int_{\mathcal{R}} \nabla \cdot F(X)\, dX = \int_{\mathcal{R}} \nabla \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} dx\, dy\, dz = \int_{\mathcal{R}} 3\, dx\, dy\, dz$$

$$= 3 \int_{\mathcal{R}} dx\, dy\, dz = 3 \cdot \frac{4}{3}\pi = 4\pi.$$

The Divergence Theorem equates two different integrals; one over the region $\mathcal{R}$ and one over Σ, the boundary of $\mathcal{R}$. It can happen that one or the other of these integrals is substantially easier to calculate than the other, so it may be advantageous to switch from one integral to the other to facilitate computation.

Example 2 Let $\mathcal{R}$ and Σ be as in the preceding example, but now

$$F(X) = F(x, y, z) = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}.$$

Then

$$F(X) \cdot \nu(X) = \frac{x^4 + y^4 + z^4}{\sqrt{x^2 + y^2 + z^2}}.$$

In this case the surface integral,

$$\int_\Sigma \frac{x^4 + y^4 + z^4}{\sqrt{x^2 + y^2 + z^2}}\, d\sigma$$

is not easy to compute. But $\nabla \cdot F(x, y, z) = 3\left(x^2 + y^2 + z^2\right)$ and thus, using spherical coordinates,

$$\int_\mathcal{R} \nabla \cdot F(x, y, z)\, dx\, dy\, dz = \int_0^{2\pi} \int_0^{\pi} \int_0^1 3r^2\, r^2 \sin\phi\, dr\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi} \sin\phi\, d\phi \int_0^1 3r^4\, dr = 12\pi \int_0^1 r^4\, dr = 12\pi \left.\frac{r^5}{5}\right|_0^1 = \frac{12\pi}{5}.$$

Figure 1: Regions for Gauss Divergence Theorem