Surfaces in Space; Surface Integrals

Various Representations of Surfaces

As studied here, a *surface* can be thought of as a two-dimensional *sheet* Σ (the more technical term is *manifold*) lying in three dimensional space $\mathbf{R}^3$. There are a number of ways in which such surfaces may be represented for analytical purposes.

i) The form $\phi(X) = 0$. This might also be called an *implicit* representation of the surface because it does not specify any variable directly in terms of others.

Example 1 The sphere of radius $r > 0$, Σ_r, may be represented as

$$\Sigma_r = \left\{ (x,y,z) \in \mathbf{R}^3 \,\middle|\, x^2 + y^2 + z^2 = r^2 \right\}.$$

Explicit Representation If we can solve for one of the variables in terms of the others, we obtain an *explicit* representation.

Example 2 The sphere Σ_r of Example 1 can also be represented as

$$\left\{ (x,y,z) \in \mathbf{R}^3 \,\middle|\, x = \pm\sqrt{r^2 - y^2 - z^2}, \text{ or } y = \pm\sqrt{r^2 - x^2 - z^2} \right\}.$$

Parametric Representation If we express all 3 variables in terms of 2 parameters, we have a *parametric* representation of the surface.

Example 3 We may represent the sphere Σ_r in Examples 1,2 in terms of "spherical coordinates":

$$\Sigma_r = \left\{ (x,y,z) \in \mathbf{R}^3 \,\middle|\, \begin{array}{c} x = r\cos\theta\sin\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{array}, \begin{array}{c} 0 \le \theta < 2\pi, \\ 0 \le \phi \le \pi \end{array} \right\}.$$

Here the spherical coordinate parameters θ and ϕ are used to *generate* the surface.

The Element of Surface Area for Explicit Representation Suppose the surface Σ under discussion is represented as $\Sigma : z = f(x,y)$, with $f(x,y)$ continuously differentiable in some region of $\mathbf{R}^2$. We form a small rectangle R in the (x,y) plane with vertices

$$(x_0, y_0)\,,\ (x_0 + \delta x, y_0)\,,\ (x_0, y_0 + \delta y)\,,\ (x_0 + \delta x, y_0 + \delta y)\,.$$

To obtain an approximate expression for the area of the portion of the surface Σ lying above this rectangle we use the area of the parallelogram in R^3 generated by the vectors $B - A$ and $C - A$, where

$$A = (x_0, y_0, f(x_0, y_0));\ \ B = (x_0 + \delta x, y_0, f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\,\delta x);$$

$$C = (x_0, y_0 + \delta y, f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0)\,\delta y).$$

This area is $\quad \|(B - A) \times (C - A)\| =$

$$\left\| \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \delta x & 0 & \frac{\partial f}{\partial x}\,\delta x \\ 0 & \delta y & \frac{\partial f}{\partial y}\,\delta y \end{pmatrix} \right\| = \left\| \mathbf{i} \left(-\frac{\partial f}{\partial x}\,\delta x\,\delta y \right) - \mathbf{j} \left(\frac{\partial f}{\partial y}\delta x\,\delta y \right) + \mathbf{k}\,\delta x\,\delta y \right\|$$

$$= \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ \delta x\,\delta y.$$

Repeating this operation for a large number of rectangles filling a region $\mathcal{R} \subset R^2$ and then letting the maximum diameter of the rectangles tend to 0 (which means we must let the number of rectangles tend to ∞, of course), we obtain the area of the portion of the surface above $\mathcal{R}$ as

$$\text{Area}\,(\Sigma) = \int_{\mathcal{R}} \sqrt{1 + \left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2}\ dx\,dy.$$

Surface Area for Two Dimensional Parametric Surfaces

Suppose we have a 2-dimensional surface $\Sigma \subset R^3$ which is generated parametrically by equations of the form

$$\Sigma : \; X(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}, \;\; (s,t) \in S,$$

where $x(s,t)$, $y(s,t)$ and $z(s,t)$ are continuously differentiable in a region S of the two dimensional parameter space with coordinates s and t. We want to be able to express surface area on Σ in terms of the standard element of area, $ds\,dt$, in S. To this end consider a small rectangle in S with corners (s_0,t_0), (s_0+ds,t_0), (s_0,t_0+dt) and (s_0+ds,t_0+dt). We let $X_0 = X(s_0,t_0)$. Then, to first order in ds and dt we have

$$X(s_0 + ds, t_0) \simeq X_0 + \frac{\partial X}{\partial s}(s_0,t_0)\,ds \equiv X_1;$$

$$X(s_0, t_0 + dt) \simeq X_0 + \frac{\partial X}{\partial t}(s_0,t_0)\,dt \equiv X_2.$$

From the differentiability of $X(s,t)$ we see that the small rectangle in S just described is carried, via $X(s,t)$, into a portion of the surface Σ which is closely approximated by the parallelogram with corners X_0, X_1, X_2 and

$$X_3 \equiv X_0 + \frac{\partial X}{\partial s}(s_0,t_0)\,ds + \frac{\partial X}{\partial t}(s_0,t_0)\,dt.$$

The area of this parallelogram can be expressed in terms of the vectors

$$\frac{\partial X}{\partial s}(s_0,t_0)\,ds = \begin{pmatrix} \frac{\partial x}{\partial s}(s_0,t_0)\,ds \\ \frac{\partial y}{\partial s}(s_0,t_0)\,ds \\ \frac{\partial z}{\partial s}(s_0,t_0)\,ds \end{pmatrix} \text{ and } \frac{\partial X}{\partial t}(s_0,t_0)\,dt = \begin{pmatrix} \frac{\partial x}{\partial t}(s_0,t_0)\,dt \\ \frac{\partial y}{\partial t}(s_0,t_0)\,dt \\ \frac{\partial z}{\partial t}(s_0,t_0)\,dt \end{pmatrix}.$$

From our earlier study of the cross product we know that the area of this parallelogram is $\left\| \frac{\partial X}{\partial s}(s_0,t_0)\,ds \times \frac{\partial X}{\partial t}(s_0,t_0)\,dt \right\|$. Generalizing this result to arbitrary points (s_0, t_0) and very small rectangles we see that if we let $d\Sigma$ denote the element of surface area on Σ we have

$$d\Sigma = \left\| \frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t} \right\| ds\, dt.$$

In general this cross product is rather complicated but in particular cases it can be quite easy to express.

Example 4 As we have seen previously, the sphere of radius r, centered at the origin in R^3, can be expressed parametrically by

$$X = X(\theta, \phi) = \begin{pmatrix} r \cos\theta \sin\phi \\ r \sin\theta \sin\phi \\ r \cos\phi \end{pmatrix}.$$

Then

$$\frac{\partial X}{\partial \theta} = \begin{pmatrix} -r \sin\theta \sin\phi \\ r \cos\theta \sin\phi \\ 0 \end{pmatrix}, \quad \frac{\partial X}{\partial \phi} = \begin{pmatrix} r \cos\theta \cos\phi \\ r \sin\theta \cos\phi \\ -r \sin\phi \end{pmatrix}$$

and thus, in this case $d\Sigma = \|\det \mathcal{M}\|\; d\theta\, d\phi$, where $\mathcal{M}$ is the symbolic matrix

$$\mathcal{M} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin\theta \sin\phi & r \cos\theta \sin\phi & 0 \\ r \cos\theta \cos\phi & r \sin\theta \cos\phi & -r \sin\phi \end{pmatrix}.$$

With some computation we can see that the norm of this cross product reduces to the value $r^2 \sin\phi$ and thus here we have

$$d\Sigma = r^2 \sin\phi\, d\theta\, d\phi.$$

The earlier case of a surface given explicitly by $z = f(x, y)$ is just a special case of the result just described wherein the coordinates x and y are chosen as the parameters, so that

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix}$$

and

$$\frac{\partial X}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{pmatrix}, \quad \frac{\partial X}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

and therefore the matrix $\mathcal{M}$ becomes

$$\mathcal{M} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{pmatrix} = -\mathbf{i}\,\frac{\partial f}{\partial x} - \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}$$

with norm $\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}$ so that

$$d\Sigma = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$$

in agreement with our earlier result.

Example 5 Consider the paraboloid $z = a\,(x^2 + y^2) = f(x, y)$. Here $\frac{\partial f}{\partial x} = 2ax$, $\frac{\partial f}{\partial y} = 2ay$, so

$$d\Sigma = \sqrt{1 + (2ax)^2 + (2ay)^2}\,dx\,dy = \sqrt{1 + 4a^2\,(x^2 + y^2)}\,dx\,dy.$$

Definitions of Surface Integrals

Explicit Representation If Σ is a surface given explicitly by $z = f(x, y)$, where $f(x, y)$ is continuously differentiable for $(x, y) \in \mathcal{R} \subset R^2$, and if $h(x, y, z)$ is continuous in a region of R^3 which includes the surface Σ, then

$$\int_\Sigma h(x, y, z)\,d\Sigma = \int_\mathcal{R} h(x, y, f(x, y))\,d\Sigma$$

$$= \int_\mathcal{R} h(x, y, f(x, y))\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy.$$

Parametric Representation If Σ is given parametrically by

$$\Sigma : \; X(s,t) \;=\; \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}, \quad (s,t) \in S,$$

then

$$\begin{aligned} \int_\Sigma h(x,y,z)\, d\Sigma \;&=\; \int_\Sigma h(x(s,t), y(s,t), z(s,t))\, d\Sigma \\ =\; &\int_S h(x(s,t), y(s,t), z(s,t)) \left\| \frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t} \right\| \, ds\, dt. \end{aligned}$$

Here we have, in general, $\frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t} \;=$

$$\mathbf{i}\left(\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} \;-\; \frac{\partial z}{\partial s}\frac{\partial y}{\partial t}\right) - \mathbf{j}\left(\frac{\partial x}{\partial s}\frac{\partial z}{\partial t} \;-\; \frac{\partial z}{\partial s}\frac{\partial x}{\partial t}\right) + \mathbf{k}\left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} \;-\; \frac{\partial y}{\partial s}\frac{\partial x}{\partial t}\right).$$

Example 6: Area of a Sphere of Radius r Let Σ_r be the sphere of radius r, centered at the origin, parametrized by the angles θ and ϕ. Then

$$\begin{aligned} \text{Area } (\Sigma_r) \;&=\; \int_{\Sigma_r} ds \;=\; \int_0^\pi \int_0^{2\pi} r^2 \sin\phi \, d\theta\, d\phi \\ &=\; 2\pi r^2 \int_0^\pi \sin\phi\, d\phi \;=\; 4\pi r^2. \end{aligned}$$

Example 7 Consider a hemispherical shell whose mid-surface is given by

$$\Sigma_a : z \;=\; f(x,y) \;=\; \sqrt{a^2 \;-\; x^2 \;-\; y^2}.$$

Assume also that the thickness of the shell is proportional to its height above the (x,y) plane, i.e., it has the form $b\sqrt{a^2 \;-\; x^2 \;-\; y^2}$ for some constant b. We compute the volume of the shell as

$$b\int_{\Sigma_a} \sqrt{a^2 - x^2 - y^2}\, ds = b\int_{\mathcal{R}_a} \sqrt{a^2 - x^2 - y^2} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy,$$

where $\mathcal{R}_a$ is the disk of radius a centered at the origin in $\mathbf{R}^2$.

Since

$$1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} = \frac{a^2}{a^2 - x^2 - y^2},$$

we have

$$\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

and the integral becomes

$$b \int_{\mathcal{R}_a} a \, dx \, dy = b \int_S \sqrt{a^2 - x^2 - y^2} \, \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dx \, dy$$

$$= ba \int_{\mathcal{R}_a} dx \, dy = ba \, \pi a^2 = \pi \, b \, a^3.$$