Differential Operators on Vector Fields; the Divergence and the Curl

The Divergence Suppose we have a vector field in $\mathbf{R}^n$:

$$Y \,=\, F(X), \text{ i.e., } \begin{array}{rcl} y_1 &=& f_1(x_1, x_2, ..., x_n) \\ y_2 &=& f_2(x_1, x_2, ..., x_n) \\ & \vdots & \\ y_n &=& f_n(x_1, x_2, ..., x_n) \end{array} .$$

Assuming the vector field to be continuously differentiable, the **divergence** of $F(X)$, written div $F(X)$, or $\nabla^* F(X)$, is the continuous scalar valued function

$$\begin{aligned} \nabla^* F(X) \,=\, &= \frac{\partial f_1}{\partial x_1}(X) \,+\, \frac{\partial f_2}{\partial x_2}(X) \,+\, ... \,+\, \frac{\partial f_n}{\partial x_n}(X) \\ &= \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_k}\,(x_1, x_2, ..., x_n)\,. \end{aligned}$$

As such, we see that the divergence is defined for a vector field of *any dimension.*

Examples 1 For the field in R^2,

$$F(X) \,=\, \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \,=\, \begin{pmatrix} 2xy \\ x^2 \,+\, y^2 \end{pmatrix},$$

we have

$$\nabla^* F(X) \,=\, \frac{\partial(2xy)}{\partial x} \,+\, \frac{\partial(x^2+y^2)}{\partial y} \,=\, 2y \,+\, 2y \,=\, 4y.$$

For the field in R^3,

$$F(X) \,=\, \begin{pmatrix} x/\sqrt{x^2+y^2+z^2} \\ y/\sqrt{x^2+y^2+z^2} \\ z/\sqrt{x^2+y^2+z^2} \end{pmatrix}$$

we note that $\frac{\partial}{\partial x}\left(x/\sqrt{x^2+y^2+z^2}\right)$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}} = \frac{y^2\,+\,z^2}{(x^2+y^2+z^2)^{3/2}},$$

and, similarly,

$$\frac{\partial}{\partial y}\left(y/\sqrt{x^2+y^2+z^2}\right) = \frac{x^2\,+\,z^2}{(x^2+y^2+z^2)^{3/2}},$$

$$\frac{\partial}{\partial z}\left(z/\sqrt{x^2+y^2+z^2}\right) = \frac{x^2\,+\,y^2}{(x^2+y^2+z^2)^{3/2}},$$

from which we see that

$$\nabla^* \begin{pmatrix} x/\sqrt{x^2+y^2+z^2} \\ y/\sqrt{x^2+y^2+z^2} \\ z/\sqrt{x^2+y^2+z^2} \end{pmatrix} = \frac{2}{(x^2+y^2+z^2)^{1/2}}.$$

The Curl The **curl** is defined only for continuously differentiable **three dimensional** vector fields, though there is a modification which is useful in the two dimensional case. If we have

$$F(X) = \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{pmatrix},$$

then **curl** $F(X) \equiv (\nabla \times F)(X)$ is **another three dimensional field**, defined by the symbolic determinant

$$(\nabla \times F)(X) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(X) & g(X) & h(X) \end{pmatrix}$$

$$= \mathbf{i}\left(\frac{\partial h}{\partial y}(X) - \frac{\partial g}{\partial x}(X)\right) - \mathbf{j}\left(\frac{\partial h}{\partial x}(X) - \frac{\partial f}{\partial z}(X)\right) + \mathbf{k}\left(\frac{\partial g}{\partial x}(X) - \frac{\partial f}{\partial y}(X)\right).$$

Example 2 Let

$$F(X) = \begin{pmatrix} y + 2z \\ x + 2z \\ x + 2y \end{pmatrix}.$$

Then the curl is

$$(\nabla \times F)(X) = \mathbf{i}\,(2 - 2) - \mathbf{j}\,(1 - 2) + \mathbf{k}\,(1 - 1) = \mathbf{j}.$$

If we have a continuously differentiable two dimensional field $F(x, y) = \mathbf{i}\ f(x, y) + \mathbf{j}\ g(x, y)$, adjoining a 0 third component we obtain the three dimensional field

$$\hat{F}(X) = \begin{pmatrix} f(x, y) \\ g(x, y) \\ 0 \end{pmatrix}$$

for which we readily see that the curl is

$$\left(\nabla \times \hat{F}\right)(X) = \mathbf{k}\left(\frac{\partial g}{\partial x}(X) - \frac{\partial f}{\partial y}(X)\right).$$

Since $\mathbf{k}$ can be assumed here, we regard the curl of the two dimensional vector field $F(x, y)$ as being the scalar $\frac{\partial g}{\partial x}(X) - \frac{\partial f}{\partial y}(X)$. Thus, for example,

$$\mathbf{curl}\begin{pmatrix} -y \\ x \end{pmatrix} = \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} = 2.$$

Since the two dimensional curl is always scalar, we will not use the symbol $\nabla \times F$ when referring to it.

Proposition If $\phi(X) = \phi(x_1, x_2, ..., x_n)$ is a twice continuously differentiable function and $F(X)$ is the gradient field

$$F(X) = \nabla\phi(X),$$

then

$$(\nabla \times \nabla\phi)(X) \equiv 0.$$

Proof This is a simple computation:

$$\textbf{curl}\ (\nabla\phi) = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x}(X) & \frac{\partial\phi}{\partial y}(X) & \frac{\partial\phi}{\partial z}(X) \end{pmatrix}$$

$$= \mathbf{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \mathbf{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \mathbf{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = 0,$$

the last equality following from the familiar equality of mixed partial derivatives.

Some important **differential operators** can be constructed by composition of the simple operators introduced here. For example, if $\phi(x_1, x_2, ..., x_n)$ is twice continuously differentiable we can construct

$$\nabla^*\nabla\phi(X) = \nabla^*\left(\textbf{grad}\,\phi\right)(X) = \nabla^*\begin{pmatrix} \frac{\partial\phi}{\partial x_1} \\ \frac{\partial\phi}{\partial x_2} \\ \vdots \\ \frac{\partial\phi}{\partial x_n} \end{pmatrix} = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + ... + \frac{\partial^2\phi}{\partial x_n^2}.$$

This **second order differential operator** is called the **Laplacian** of the function $\phi(X)$. For this operator the notation $\nabla^2\,\phi$ is often used.

Example 3 If $\phi(x, y) = e^{-x}\sin y$ we have

$$\nabla^2\,\phi(x, y) = \frac{\partial^2 e^{-x}\sin y}{\partial x^2} + \frac{\partial^2 e^{-x}\sin y}{\partial y^2} = e^{-x}\sin y - e^{-x}\sin y = 0.$$

On the other hand, for $\phi(x, y, z) = x^2 + y^2 + z^2$ we clearly have $\nabla^2\,\phi(x, y, z) = 6$. A function $\phi(x_1, x_2, ..., x_n)$ whose Laplacian is identically equal to zero is called a **harmonic function**. Such functions are of exceptional importance in electromagnetism, elasticity, fluid flow and many other applications.