Changing Variables in Multiple Integrals

The One-Dimensional Case Here it is easy to recall the procedure presented in the standard calculus. Given an integral

$$\int_a^b \phi(x)\,dx,$$

and a differentiable, single-valued (i.e., differentiable and either monotone increasing or monotone decreasing) function $f(t)$ whose range includes the interval $[a,b]$, we can introduce the change of variable $x = f(t)$ and, with $a = t_a,\ b = t_b$, write

$$\int_a^b \phi(x)\,dx = \int_{t_a}^{t_b} \phi(f(t))\,\frac{df}{dt}(t)\,dt.$$

In fact, $f(t)$ does not have to be monotone; e.g., if there is a value c with $c < a$, and $\phi(x)$ is defined on the extended interval $[c,b]$, and if there is a value t_c with $t_a < t_c < t_b$ such that $f(t)$ decreases from the value a to the value c on $[t_a,t_c]$ and then increases from the value c to the value b on $[t_c,t_b]$, then

$$\int_a^b \phi(x)\,dx = \int_a^c \phi(x)\,dx + \int_c^b \phi(x)\,dx$$

$$= \int_{t_a}^{t_c} \phi(f(t))\frac{df}{dt}(t)\,dt + \int_{t_c}^{t_b} \phi(f(t))\frac{df}{dt}(t)\,dt = \int_{t_a}^{t_b} \phi(f(t))\frac{df}{dt}(t)\,dt.$$

Example Compute $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. The standard approach here would be to let $x = f(\theta) = \sin\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$, where $\sin\theta$ is monotone increasing, and compute

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2\theta}}\frac{d}{d\theta}\sin\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta}\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}.$$

On the interval $-\pi \leq \theta \leq \frac{\pi}{2}$ we also have $\sin(-\pi) = 0, \ \sin(\frac{\pi}{2}) = 1$, but $\sin\theta$ is not monotone there. Nevertheless we have

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_{-\pi}^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2\theta}} \frac{d}{d\theta} \sin\theta \, d\theta$$

$$= \int_{-\pi}^{-\frac{\pi}{2}} \left(-\frac{1}{\cos\theta} \right) \cos\theta \, d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cos\theta} \cos\theta \, d\theta$$

$$= \int_{-\pi}^{-\frac{\pi}{2}} (-1) \, d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = -\frac{\pi}{2} + \pi = \frac{\pi}{2}.$$

We will see that for multiple integrals, i.e., integrals in $\mathbf{R}^n$, where $n > 1$, a number of similar considerations apply but they have to be treated in a somewhat different manner.

Change of Variables for Integrals in $\mathbf{R}^2$. Suppose $\mathcal{R}$ is a region in $\mathbf{R}^2$ and $\phi(X) = \phi(x,y)$ is a continuous, or piecewise continuous, function defined on $\mathcal{R}$. Then we can define the integral

$$\int_{\mathcal{R}} \phi(X) \, dX = \int_{\mathcal{R}} \phi(x,y) \, dx \, dy.$$

In some circumstances it is easier to evaluate, or study, the integral with the aid of a change of coordinates. We introduce a new vector variable

$$\Xi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and a vector functional relationship, expressing the change of coordinates,

$$X = F(\Xi) = \begin{pmatrix} f(\xi,\eta) \\ g(\xi,\eta) \end{pmatrix}.$$

We suppose that $F(\Xi)$ is continuously, or piecewise continuously, differentiable and that the mapping induced by this relationship is one-to-one and takes a region $\hat{\mathcal{R}}$ in the ξ, η plane onto a region $\mathcal{R}$ in the x, y plane. Then

$$\psi(\Xi) = \psi(\xi,\eta) \equiv \phi(F(\Xi)) = \phi(f(\xi,\eta), g(\xi,\eta))$$

is a continuous, or piecewise continuous, function defined on $\hat{\mathcal{R}}$. The question we want to consider is: how do we re-express $\int_{\mathcal{R}} \phi(X)\, dX$ in terms of the function $\psi(\Xi)$ and the region $\hat{\mathcal{R}}$?

If we partition the region $\hat{\mathcal{R}}$ via a set of rectangles $\{R_k \,|k\, = \, 1, 2, ..., K\}$ and, for each k, let $\Omega_k \, = \, (\omega_k, \zeta_k)$ be a point in R_k, then, given a continuous, or piecewise continuous function $\upsilon(\xi, \eta)$ defined on $\hat{\mathcal{R}}$, we have the approximation

$$\int_{\hat{\mathcal{R}}} \psi(\xi, \eta)\, d\xi\, d\eta \approx \sum_{k=1}^{K} \psi(\omega_k, \zeta_k)\, \delta\xi_k \times \delta\eta_k,$$

where the rectangle R_k has dimensions $\delta\xi_k \times \delta\eta_k$.

Since $X \, = \, F(\Xi)$ covers, as Ξ ranges over $\hat{\mathcal{R}}$, the region $\mathcal{R}$, in a one-to one manner, given a continuous, or piecewise continuous function $\psi(\Xi) \, = \, \psi(\xi, \eta)$ defined on $\hat{\mathcal{R}}$, we can define a function $\phi(X) \, = \, \phi(x, y)$ on $\mathcal{R}$ by requiring

$$\phi(F(\Xi)) \, = \, \phi(f(\xi, \eta), g(\xi, \eta)) \, = \, \psi(\Xi) \, = \, \psi(\xi, \eta), \;\; \Xi \, \in \, \hat{\mathcal{R}}.$$

Let $Q_k \, = \, F(R_k)$, i.e., Q_k is the image of the rectangle R_k under $X \, = \, F(\Xi)$, and let A_k denote the area of Q_k. The set $\{Q_k \,|k\, = \, 1, 2, ..., K\}$ forms a partition of $\mathcal{R}$, albeit a nonstandard one since the Q_k are not rectangles. Nevertheless, as we refine the partition $\{R_k \,|k\, = \, 1, 2, ..., K\}$ of $\hat{\mathcal{R}}$ by letting the maximum diameter of the R_k tend to zero, so that the sum approximating $\int_{\hat{\mathcal{R}}} \upsilon(\xi, \eta)\, d\xi\, d\eta$ more and more closely approaches that integral, the diameters of the corresponding sets $\{P_k\}$ will also tend to zero and the union of the sets Q_k will more and more closely cover the region $\mathcal{R}$ and the approximation relationship

$$\int_{\mathcal{R}} \phi(X)\, dX \approx \sum_{k=1}^{K} \phi(F(\Omega_k))\, A_k$$

will become more and more accurate.

Let us choose one of the rectangles in the partition of $\hat{\mathcal{R}}$ and call it R, supposing it to have dimensions $d\xi \times d\eta$. Let us suppose its lower left hand corner, Ξ_0, and its other three corners, are given by

$$\Xi_0 = (\xi_0, \eta_0)\,,\ (\xi_0 + \delta\xi, \eta_0)\,,\ (\xi_0, \eta_0 + \delta\eta)\,,\ (\xi_0 + \delta\xi, \eta_0 + \delta\eta)\,.$$

Letting

$$\tilde{F}(\Xi) = F(\Xi_0) + \nabla F(\Xi_0)(\Xi - \Xi_0)$$

be the linear approximation to $F(\Xi)$ based at the point Ξ_0, the set $Q = F(R)$ can be approximated by $P = \tilde{F}(R)$. Because $\tilde{F}(\Xi)$ is linear in Ξ it is easy to see that P is a parallelogram. Three of the corners, or vertices, of this parallelogram are

$$\tilde{F}\,(\xi_0, \eta_0) = F(\Xi_0),$$

$$\tilde{F}\,(\xi_0 + \delta\xi, \eta_0) = F(\Xi_0) + \nabla F(\Xi_0)\begin{pmatrix} \delta\xi \\ 0 \end{pmatrix} = F(\Xi_0) + \begin{pmatrix} \frac{\partial f}{\partial \xi}(\Xi_0)\,\delta\xi \\ \frac{\partial g}{\partial \xi}(\Xi_0)\,\delta\xi \end{pmatrix}$$

and

$$F\,(\xi_0, \eta_0 + \delta\eta) = = F(\Xi_0) + \nabla F(\Xi_0)\begin{pmatrix} 0 \\ \delta\eta \end{pmatrix} = F(\Xi_0) + \begin{pmatrix} \frac{\partial f}{\partial \eta}(\Xi_0)\,\delta\eta \\ \frac{\partial g}{\partial \eta}(\Xi_0)\,\delta\eta \end{pmatrix}.$$

The area of this parallelogram is then

$$A = \left\| \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial \xi}(\Xi_0)\delta\xi & \frac{\partial g}{\partial \xi}(\Xi_0)\delta\xi & 0 \\ \frac{\partial f}{\partial \eta}(\Xi_0)\delta\eta & \frac{\partial g}{\partial \eta}(\Xi_0)\delta\eta & 0 \end{pmatrix} \right\| = \delta\xi\delta\eta \left| \det \begin{pmatrix} \frac{\partial f}{\partial \xi}(\Xi_0) & \frac{\partial g}{\partial \xi}(\Xi_0) \\ \frac{\partial f}{\partial \eta}(\Xi_0) & \frac{\partial g}{\partial \eta}(\Xi_0) \end{pmatrix} \right|$$

$$= \delta\xi\delta\eta \left| \det \begin{pmatrix} \frac{\partial f}{\partial \xi}(\Xi_0) & \frac{\partial f}{\partial \eta}(\Xi_0) \\ \frac{\partial g}{\partial \xi}(\Xi_0) & \frac{\partial g}{\partial \eta}(\Xi_0) \end{pmatrix} \right| = \delta\xi\delta\eta\ \left|\det \nabla F(\Xi_0)\right|.$$

We repeat this discussion for each of the rectangles R_k partitioning $\hat{\mathcal{R}}$, in each case letting Ξ_k be the lower left hand corner of R_k, letting the dimensions of R_k be $\delta\xi_k \times \delta\eta_k$ and letting Ω_k be an arbitrary point in R_k. The approximation relationship described earlier then may be replaced by

$$\int_{\mathcal{R}} \phi(X)\,dX \approx \sum_{k=1}^{K} \phi(F(\Omega_k))\,A_k$$
$$= \sum_{k=1}^{K} \phi(F(\Omega_k))\,|\det \nabla F(\Xi_k)|\ \delta\xi_k \times \delta\eta_k$$
$$= \sum_{k=1}^{K} \psi(\omega_k,\zeta_k)\,|\det \nabla F(\Xi_k)|\ \delta\xi_k \times \delta\eta_k.$$

Comparing this with the earlier approximation relationship for the integral of an arbitrary $\psi(\xi,\eta)$ over $\hat{\mathcal{R}}$, i.e.,

$$\int_{\hat{\mathcal{R}}} \psi(\xi,\eta)\,d\xi\,d\eta \approx \sum_{k=1}^{K} \psi(\omega_k,\zeta_k)\,\delta\xi_k \times \delta\eta_k,$$

we see that in the limit, as we let the maximum diameter of the R_k tend to zero, we obtain the relationship

$$\int_{\mathcal{R}} \phi(x,y)\,dx\,dy \;=\; \int_{\mathcal{R}} \phi(X)\,dX \;=\; \int_{\hat{\mathcal{R}}} \phi(F(\Xi))\ |\det \nabla F(\Xi)|\ d\Xi$$
$$= \int_{\hat{\mathcal{R}}} \phi(f(\xi,\eta),g(\xi,\eta)) \left|\det \begin{pmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \eta} \\ \frac{\partial g}{\partial \xi} & \frac{\partial g}{\partial \eta} \end{pmatrix} (\xi,\eta)\right| \ d\xi\,d\eta.$$

Example 1 Let $\mathcal{R}$ be the disk of radius ρ centered at the origin in the x, y plane:

$$\mathcal{R} \;=\; \left\{x, y \,\middle|\, x^2 + y^2 \;\leq\; \rho\right\},$$

and let $\phi(x,y) \;=\; e^{(x^2+y^2)}$. We wish to compute

$$\int_{\mathcal{R}} e^{-(x^2+y^2)}\,dx\,dy.$$

For this purpose the geometry and the form of the integrand both suggest the use of polar coordinates

$$r \;=\; \sqrt{x^2+y^2},\ \ \theta \;=\; \tan^{-1}\frac{y}{x}.$$

Accordingly we introduce the change of coordinates, expressed by

$$x = f(r,\theta) = r\,\cos\theta,\;\; y = g(r,\theta) = r\,\sin\theta.$$

The variables r and θ fill the roles of ξ and η in our earlier general discussion. The "mapping" described by $F = (f,g)^*$ carries the rectangle

$$\hat{\mathcal{R}} = \{r,\theta\,|0 < r \le \rho,\;\; 0 < \theta \le 2\pi\}$$

in the r, θ plane onto the disk $\mathcal{R}$.

Here we have

$$\left|\det\begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta} \end{pmatrix}\right| = \left|\det\begin{pmatrix} \cos\theta & -r\,\cos\theta \\ \sin\theta & r\,\cos\theta \end{pmatrix}\right|$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Thus the area elements are related by $dx\,dy = r\,dr\,d\theta$ and we have

$$\int_{\mathcal{R}} e^{-(x^2+y^2)}\,dx\,dy = \int_{\hat{\mathcal{R}}} e^{-r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\rho} e^{-r^2} r\,dr\,d\theta$$

$$= 2\pi\int_0^{\rho} e^{-r^2} r\,dr = \pi\int_0^{\rho}\frac{d}{dr}\left(-e^{-r^2}\right)dr = \pi\left(1 - e^{-\rho^2}\right).$$

An immediate corollary, as we let $\rho \to \infty$, is

$$\int_{R^2} e^{-(x^2+y^2)}\,dx\,dy = \pi.$$

Change of Variables for Integrals in $\mathbf{R}^3$, $\mathbf{R}^n$. In three dimensions and, for that matter, in $\mathbf{R}^n$ for any integer $n \ge 3$, changing coordinates in integrals works out in much the same way as in $\mathbf{R}^2$. Here we will content ourselves with describing what modifications are required in the three dimensional, as compared with the two dimensional, case. Thus we suppose we have a continuous, or piecewise continuous function $\phi(X) = \phi(x,y,z)$ defined on a region $\mathcal{R} \subset \mathbf{R}^3$ and we assume there is a one-to-one, continuously, or piecewise continuously, differentiable vector function, or "map"

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X = F(\Xi) = \begin{pmatrix} f(\xi,\eta,\zeta) \\ g(\xi,\eta,\zeta) \\ h(\xi,\eta,\zeta) \end{pmatrix}$$

carrying another three dimensional region $\hat{\mathcal{R}} \subset \hat{\mathbf{R}}^3$ onto $\mathcal{R}$. We introduce rectangles R_k, now three dimensional ones, partitioning $\hat{\mathcal{R}}$, let $Q_k = F(R_k)$ be the image regions partitioning $\mathcal{R}$ and let $P_k = \tilde{F}(R_k)$ be the approximating parallelopipedons, just as in the two dimensional case, where $\tilde{F}$ is a certain linear approximation to F. More specifically, selecting a typical R_k and calling it R, we select that corner, or vertex of R corresponding to the least values of ξ, η and ζ and call that vertex Ξ_0. Then, on R,

$$\tilde{F}(\Xi) = F(\Xi_0) + \nabla F(\Xi_0)(\Xi - \Xi_0), \nabla F(\Xi_0) = \begin{pmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \eta} & \frac{\partial f}{\partial \zeta} \\ \frac{\partial g}{\partial \xi} & \frac{\partial g}{\partial \eta} & \frac{\partial g}{\partial \zeta} \\ \frac{\partial h}{\partial \xi} & \frac{\partial h}{\partial \eta} & \frac{\partial h}{\partial \zeta} \end{pmatrix} (\xi_0, \eta_0, \zeta_0) .$$

The parallelopipedon $P = \tilde{F}(R)$ has eight vertices. Only four of these concern us:

$$X_0 = F(\Xi_0) = \tilde{F}(\Xi_0),$$

$$X_1 \equiv X_0 + \nabla F(\Xi_0) \begin{pmatrix} \delta\xi \\ 0 \\ 0 \end{pmatrix} = X_0 + \left(\frac{\partial f}{\partial \xi}\,\delta\xi,\ \frac{\partial g}{\partial \xi}\,\delta\xi,\ \frac{\partial h}{\partial \xi}\,\delta\xi \right)^*,$$

$$X_2 \equiv X_0 + \nabla F(\Xi_0) \begin{pmatrix} 0 \\ \delta\eta \\ 0 \end{pmatrix} = X_0 + \left(\frac{\partial f}{\partial \eta}\,\delta\eta,\ \frac{\partial g}{\partial \eta}\,\delta\eta,\ \frac{\partial h}{\partial \eta}\,\delta\eta \right)^*,$$

$$X_3 \equiv X_0 + \nabla F(\Xi_0) \begin{pmatrix} 0 \\ 0 \\ \delta\zeta \end{pmatrix} = X_0 + \left(\frac{\partial f}{\partial \zeta}\,\delta\zeta,\ \frac{\partial g}{\partial \zeta}\,\delta\zeta,\ \frac{\partial h}{\partial \zeta}\,\delta\zeta \right)^*,$$

where $\delta\xi$, $\delta\eta$ and $\delta\zeta$ are the (positive) side lengths of R. The volume of the parallelogram $P = \tilde{F}(R)$ spanned by these vertices is then given by the *vector triple product*

$$\begin{aligned} V(P) &= |(X_1 - X_0) * (X_2 - X_0) \times (X_3 - X_0)| \\ &= \left| \det \begin{pmatrix} \frac{\partial f}{\partial \xi}\delta\xi & \frac{\partial g}{\partial \xi}\delta\xi & \frac{\partial h}{\partial \xi}\delta\xi \\ \frac{\partial f}{\partial \eta}\delta\eta & \frac{\partial g}{\partial \eta}\delta\eta & \frac{\partial h}{\partial \eta}\delta\eta \\ \frac{\partial f}{\partial \zeta}\delta\zeta & \frac{\partial g}{\partial \zeta}\delta\zeta & \frac{\partial h}{\partial \zeta}\delta\zeta \end{pmatrix} \right|. \end{aligned}$$

The partial derivatives here are all evaluated at Ξ_0. Since multiplication of a row of a matrix by a constant multiplies the determinant by that constant and since the determinant of a matrix is equal to the determinant of its transpose, we have

$$V(P) = \left| \det \begin{pmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \eta} & \frac{\partial f}{\partial \zeta} \\ \frac{\partial g}{\partial \xi} & \frac{\partial g}{\partial \eta} & \frac{\partial g}{\partial \zeta} \\ \frac{\partial h}{\partial \xi} & \frac{\partial h}{\partial \eta} & \frac{\partial h}{\partial \zeta} \end{pmatrix} \right| \delta\xi\, \delta\eta\, \delta\zeta.$$

Since $\delta\xi\, \delta\eta\, \delta\zeta = V(R)$, the volume of the rectangle R, we have

$$V(P) = |\det \nabla F(\Xi_0)|\; V(R).$$

Applying this result to all of the three dimensional rectangles R_k and using the same arguments as in the two dimensional case, we again have

$$\int_{\mathcal{R}} \phi(X)\, dX = \int_{\hat{\mathcal{R}}} \phi(F(\Xi))\, |\det \nabla F(\Xi)|\; d\Xi.$$

Example 2 Let $\mathcal{R}$ be the ball of radius ρ in $\mathbf{R}^3$. We compute

$$\int_{\mathcal{R}} \log \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz.$$

Here we use the standard *spherical coordinates* r, θ, φ, the latter, with range $0 \le \varphi \le \pi$, being the angle between the half line joining the origin to the point in question and the half line $x = y = 0$, i.e., the positive z axis. The relationship between the standard cartesian coordinates x, y, z and the spherical coordinates is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = F(r,\theta,\varphi) = \begin{pmatrix} f(r,\theta,\varphi) \\ g(r,\theta,\varphi) \\ h(r,\theta,\varphi) \end{pmatrix} = \begin{pmatrix} r\,\cos\theta\,\sin\varphi \\ r\,\sin\theta\,\sin\varphi \\ r\,\cos\varphi \end{pmatrix}.$$

The Jacobian determinant of F in this case is

$$\det\begin{pmatrix} \cos\theta\,\sin\varphi & -r\,\sin\theta\,\sin\varphi & r\,\cos\theta\,\cos\varphi \\ \sin\theta\,\sin\varphi & r\,\cos\theta\,\sin\varphi & r\,\sin\theta\,\cos\varphi \\ \cos\varphi & 0 & -r\,\sin\varphi \end{pmatrix}$$

$$= r^2\sin\varphi\det\begin{pmatrix} \cos\theta\,\sin\varphi & -\sin\theta & \cos\theta\,\cos\varphi \\ \sin\theta\,\sin\varphi & \cos\theta & \sin\theta\,\cos\varphi \\ \cos\varphi & 0 & -\sin\varphi \end{pmatrix}$$

$$= r^2\sin\varphi\Big(\cos^2\varphi(-\sin^2\theta-\cos^2\theta) - \sin^2\varphi(\cos^2\theta+\sin^2\theta)\Big) = -r^2\sin\varphi.$$

Since it is the absolute value of the determinant which enters into the change of coordinates, and since $\sin\varphi \geq 0$ over the range indicated, we have

$$\int_{\mathcal{R}} \log\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{\rho} \log r\,r^2\sin\varphi\,dr\,d\theta\,d\varphi$$

$$= \frac{4\pi}{9}\int_0^{\rho}\log(r^3)\,\frac{d}{dr}(r^3)\,dr = \frac{4\pi}{9}\int_0^{\rho^3}\log u\,du = \frac{4\pi}{9}\left(\rho^3\,\log\rho^3 - \rho^3\right).$$

The result in $\mathbf{R}^n$, for an arbitrary positive integer n, is identical; for all n, defining the integral as we have, the result is

$$\int_{\mathcal{R}} \phi(X)\,dX = \int_{\hat{\mathcal{R}}} \phi(F(\Xi))\,|\det\nabla F(\Xi)|\,d\Xi.$$

Curiously, perhaps, one has to be a little bit careful in applying this result to the familiar case $n = 1$. Suppose in the integral $\int_1^2 \frac{dx}{x^2}$, which is clearly positive, we set $x = \frac{1}{\xi}$. Then $|\det \nabla F(\Xi)|$ reduces in this case to $\left|\frac{-1}{\xi^2}\right| = \frac{1}{\xi}^2$. There may be a temptation to apply the general rule in the following (incorrect) way:

$$\int_1^2 \frac{dx}{x^2} = \int_{\xi = 1/1}^{\xi = 1/2} \xi^2 \, \frac{1}{\xi^2} \, d\xi = \int_1^{1/2} d\xi = -\frac{1}{2}$$

– which is negative! The correct result is obtained, of course, by setting $dx = \frac{dx}{d\xi} \, d\xi = -\frac{d\xi}{\xi^2}$. The apparent inconsistency is resolved when we realize that our integrals $\int_{\mathcal{R}} \phi(X) \, dX$, $\int_{\hat{\mathcal{R}}} \phi(F(\Xi)) \, |\nabla F(\Xi)| \, d\Xi$ are always defined with respect to length, area, volume, etc., measured in the *positive* sense. An integral such as $\int_1^{1/2} d\xi$, in which the upper limit is less than the lower limit, is not defined this way. It is correct to write

$$\int_{[1,2]} \frac{dx}{x^2} = \int_{[\frac{1}{2},1]} \xi^2 \, \frac{1}{\xi^2} \, d\xi = \int_{[\frac{1}{2},1]} d\xi = \frac{1}{2}$$

because the latter integral means $\int_{\frac{1}{2}}^{1} d\xi$, not $\int_1^{\frac{1}{2}} d\xi$.