Green's Theorem in the Plane

Theorem Let $F(x,y) = (f(x,y), g(x,y))^*$ be a continuously differentiable vector field in a region $\mathcal{R} \subset R^2$ which includes a simple closed curve $\mathcal{C}$ and its interior $\mathbf{int}\,\mathcal{C}$. Then

$$\iint_{\mathbf{int}\,\mathcal{C}} \left(\frac{\partial g}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y) \right) dx\,dy = \int_{\mathcal{C}} F(X) \cdot dX,$$

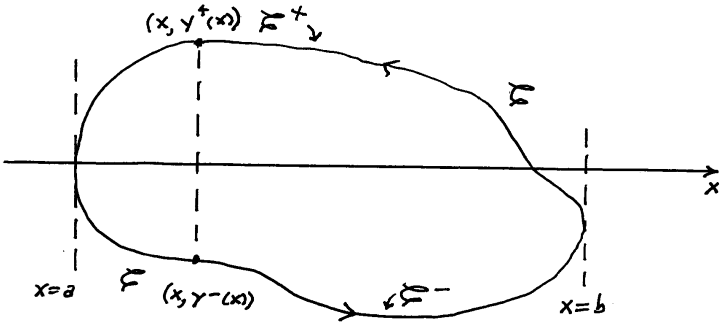
the curve $\mathcal{C}$ being oriented in the mathematically positive (counterclockwise) direction.

Remark The scalar function $\frac{\partial g}{\partial x}(X) - \frac{\partial f}{\partial y}(X)$ is the *"two - dimensional curl"* of the vector field $F(X)$. Later we will introduce the concept of curl in a more general context.

Proof We begin the case wherein the curve $\mathcal{C}$ can be decomposed into "upper" and "lower" segments, as shown in the figure, which can be parametrized in terms of x:

$$\mathcal{C}^+ : \ y = y^+(x),\ a \le x \le b,\ \mathcal{C}^- : \ y = y^-(x),\ a \le x \le b.$$

Figure 1



We will suppose the curve $\mathcal{C}$ is continuously differentiable, which here implies that the functions $y^\pm(x)$ are continuously differentiable except possibly at $x = a$ and $x = b$.

Because of the special geometry assumed, we can write the double integral of $-\frac{\partial f}{\partial y}(x,y)$ as an **iterated integral**:

$$-\iint_{\mathbf{int}\,\mathcal{C}} \frac{\partial f}{\partial y}(x,y)\,dxdy \;=\; \int_a^b \Big(-\int_{y=y^-(x)}^{y=y^+(x)} \frac{\partial f}{\partial y}(x,y)dy \Big)\,dx \;=$$

$$\int_a^b \big(-f(x,y^+(x)) \;+\; f(x,y^-(x))\big)\,dx \;=\; \int_{\mathcal{C}^+} \begin{pmatrix} f(x,y) \\ 0 \end{pmatrix} \cdot dX$$

$$+\int_{\mathcal{C}^-} \begin{pmatrix} f(x,y) \\ 0 \end{pmatrix} \cdot dX \;=\; \int_{\mathcal{C}} \begin{pmatrix} f(x,y) \\ 0 \end{pmatrix} \cdot dX$$

$$= \int_{\mathcal{C}} \begin{pmatrix} f(x,y) \\ 0 \end{pmatrix} \cdot dX,$$

because $\mathcal{C}^+$ is oriented from right to left and $\mathcal{C}^-$ is oriented from left to right in the overall counterclockwise orientation of $\mathcal{C}$.

We now further suppose that $\mathcal{C}$ can also be decomposed into simple left and right segments:

$$\mathcal{C}_r : \;\; x \;=\; x_r(y), \;\; \mathcal{C}_\ell : \;\; x \;=\; x_\ell(y),$$

each continuously differentiable except possibly at the lower and upper limits, c and d, respectively, of y. Treating the double integral of $\frac{\partial g}{\partial x}$ as we did that of $-\frac{\partial f}{\partial y}$ above, but now with reference to $\mathcal{C}_r$ and $\mathcal{C}_\ell$, we obtain

$$\iint_{\mathbf{int}\,\mathcal{C}} \frac{\partial g}{\partial x}(x,y)\,dxdy \;=\; \int_{\mathcal{C}} \begin{pmatrix} 0 \\ g(x,y) \end{pmatrix} \cdot dX.$$

Combining this with the earlier result we have the theorem as stated for curves with the simple geometry assumed.

For more complicated, but still continuously differentiable, curves, as shown in the figure, we can introduce "interior" arcs to subdivide **int** $\mathcal{C}$ into a finite number of regions, each bounded by a curve that does satisfy our earlier assumptions. If there are K such regions, bounded by curves $\mathcal{C}_k,\ \ k\ =\ 1, 2, ..., K$, we will have

$$\sum_{k\,=\,1,2,\ldots,K}\iint_{\mathbf{int}\,\mathcal{C}_k}\left(\frac{\partial g}{\partial x}(x,y)\ -\ \frac{\partial f}{\partial y}(x,y)\right)\,dxdy\ =\ \sum_{k\,=\,1,2,\ldots,K}\int_{\mathcal{C}_k}F(X){\cdot}dX.$$

The term on the left adds to the integral over $\mathbf{int}\,\mathcal{C}$. In the term on the right it will be seen that each interior arc is traversed twice, once in each direction. These integrals over interior arcs consequently cancel to zero and we are left with

$$\iint_{\mathbf{int}\,\mathcal{C}}\left(\frac{\partial g}{\partial x}(x,y)\ -\ \frac{\partial f}{\partial y}(x,y)\right)\,dx\,dy\ =\ \int_{\mathcal{C}}F(X)\cdot dX,$$

as stated in the theorem.

The requirement that the region $\mathcal{R}$ in which the field $F(X)$ is defined and continuously differentiable should include $\mathbf{int}\,\mathcal{C}$ is essential; without it the result may not hold.

Example 1 Consider the vector field

$$F(X)\ =\ \begin{pmatrix} f(x,y)\\ g(x,y)\end{pmatrix}\ =\ \begin{pmatrix}\frac{-y}{x^2+y^2}\\ \frac{x}{x^2+y^2}\end{pmatrix};\ \ \mathcal{R}\ =\ R^2\ -\ \{(0,0)\}.$$

We let $\mathcal{C}$ the the circle of radius r centered at the origin; clearly $\mathcal{R}$ fails to cover the point $(0,0)\ \in\ \mathbf{int}\,\mathcal{C}$ where the components of $F(X)$ are singular. Parametrizing $\mathcal{C}$ as $X(t)\ =\ (r\ \cos t,\ r\ \sin t)^*,\ \ 0\ \le\ t\ \le\ 2\pi$, we compute

$$\int_{\mathcal{C}}F(X){\cdot}dX\ =\ \int_0^{2\pi}\begin{pmatrix}\frac{-y}{x^2+y^2}\\ \frac{x}{x^2+y^2}\end{pmatrix}{\cdot}X'(t)\,dt\ =\ \int_0^{2\pi}\frac{1}{r}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}{\cdot}\begin{pmatrix}-r\sin t\\ r\cos t\end{pmatrix}\,dt$$

$$=\ \int_0^{2\pi}(\sin^2 t\ +\ \cos^2 t)\,dt\ =\ 2\pi.$$

But

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\ -\ \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\ =\ \frac{2}{x^2+y^2}\ -\ \frac{2(x^2+y^2)}{(x^2+y^2)^2}\ =\ 0.$$

The reason why Green's Theorem does not apply here is precisely because the region in which the field $F(X)$ is continuously differentiable does not include the point $(0,0) \in \mathbf{int}\, \mathcal{C}$.

Proposition 1 *If $\mathcal{C}$ is a simple closed curve oriented in the positive (counterclockwise) direction, then*

$$\text{Area}\,(\mathbf{int}\,\mathcal{C}) \;=\; \frac{1}{2}\int_{\mathcal{C}} \begin{pmatrix} -y \\ x \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} \;=\; \frac{1}{2}\int_{\mathcal{C}} \Big((-y)\,dx \,+\, x\,dy \Big).$$

Proof We note that the two - dimensional curl of the field $F(x,y) = (-y,x)$ is $\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} = 2$ and Green's Theorem tell us, therefore, that

$$\frac{1}{2}\iint_{\mathbf{int}\,\mathcal{C}} \mathbf{curl}\, F(x,y)\,dxdy$$

$$=\; \frac{1}{2}\iint_{\mathbf{int}\,\mathcal{C}} 2\,dxdy \;=\; \text{Area}\,(\mathbf{int}\,\mathcal{C}) \;=\; \frac{1}{2}\int_{\mathcal{C}} \Big((-y)\,dx \,+\, x\,dy \Big).$$

Example 2 Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \;=\; 1,$$

which may be parametrized as

$$x(t) \;=\; a\,\cos t,\;\; y(t) \;=\; b\,\sin t,\;\; 0 \;\le\; t \;\le\; 2\pi.$$

We compute

$$\frac{1}{2}\int_{\mathcal{C}} \begin{pmatrix} -y \\ x \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} \;=\; \int_0^{2\pi} \begin{pmatrix} -b\,\sin t \\ a\,\cos t \end{pmatrix} \cdot \begin{pmatrix} -a\sin t \\ b\,\cos t \end{pmatrix} dt$$

$$=\; \frac{1}{2}\,ab\int_0^{2\pi} (\sin^2 t \,+\, \cos^2 t)\,dt \;=\; \frac{1}{2}\,ab\int_0^{2\pi} 1\,dt \;=\; \pi\,ab.$$

The Divergence Theorem in the Plane For a continuously differentiable two - dimensional vector field $F(X) = (f(x,\, y),\, g(x,\, y))^*$ the *divergence* is defined by

$$\mathbf{div}\, F(X) = \frac{\partial f}{\partial x}(X) + \frac{\partial g}{\partial y}(X) \equiv \nabla \cdot F(X).$$

Let us consider

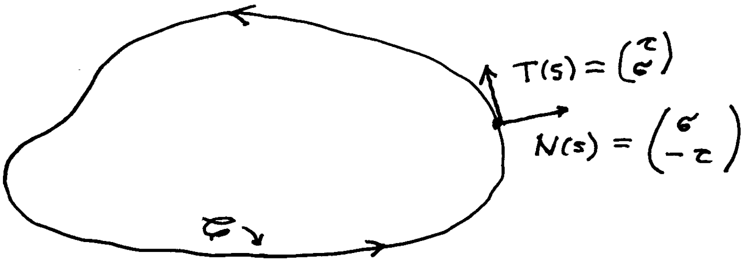
$$\hat{F}(X) = \begin{pmatrix} -g(x,y) \\ f(x,y) \end{pmatrix}.$$

Then

$$\int_{\mathbf{int}\,\mathcal{C}} \nabla \cdot F(X)\, dxdy = \int_{\mathbf{int}\,\mathcal{C}} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dxdy =$$

$$\int_{\mathbf{int}\,\mathcal{C}} \left(\frac{\partial f}{\partial x} - \frac{\partial(-g)}{\partial y} \right) dxdy = \int_{\mathcal{C}} \begin{pmatrix} -g(x,y) \\ f(x,y) \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \int_{\mathcal{C}} \hat{F}(X)^* dX.$$

Figure 2



As in the diagram, for a point (x, y) on $\mathcal{C}$ let the unit tangent vector to $\mathcal{C}$ pointing in the positive direction of $\mathcal{C}$, be $T = (\tau, \sigma)$. The unit normal vector pointing to the exterior of $\mathcal{C}$ is then $\nu = (\sigma, -\tau)$. Now, with s denoting arc length, increasing in the positive direction, on $\mathcal{C}$, we have

$$dX = \begin{pmatrix} dx \\ dy \end{pmatrix} = T(s)\, ds = \begin{pmatrix} \tau(s) \\ \sigma(s) \end{pmatrix} ds.$$

Therefore

$$\int_{\mathcal{C}} \hat{F}(X)^* dX = \int_{\mathcal{C}} \begin{pmatrix} -g(x,y) \\ f(x,y) \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \int_{\mathcal{C}} \begin{pmatrix} -g(x,y) \\ f(x,y) \end{pmatrix} \cdot \begin{pmatrix} \tau(s) \\ \sigma(s) \end{pmatrix} ds$$

$$= \int_{\mathcal{C}} \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \cdot \begin{pmatrix} \sigma(s) \\ -\tau(s) \end{pmatrix} ds \;=\; \int_{\mathcal{C}} F(X(s)) \cdot \nu(s)\, ds.$$

Thus we have

$$\int_{\mathbf{int}\,\mathcal{C}} \nabla \cdot F(X)\, dxdy \;=\; \int_{\mathcal{C}} F(X(s)) \cdot \nu(s)\, ds.$$

This is the *Gauss Divergence Theorem*; we will have a lot more to say about this later in the more general n-dimensional context.

Example 3 We observe that

$$\nabla X \;=\; \nabla \begin{pmatrix} x \\ y \end{pmatrix} \;=\; 2$$

and therefore, for an arbitrary closed curve $\mathcal{C}$,

$$\text{Area}\,(\mathbf{int}\,\mathcal{C}) \;=\; \frac{1}{2} \iint_{\mathbf{int}\,\mathcal{C}} 2\, dxdy \;=\; \frac{1}{2} \iint_{\mathbf{int}\,\mathcal{C}} \nabla \cdot X\, dxdy \;=\; \frac{1}{2} \int_{\mathcal{C}} X \cdot \nu\, ds.$$

In particular, if $\mathcal{C}$ is the circle of radius r about the origin in R^2, we have

$$\frac{1}{2} \int_{\mathcal{C}} X \cdot \nu\, ds \;=\; \int_{\mathcal{C}} \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -\sigma(s) \\ \tau(s) \end{pmatrix} ds$$

$$= \frac{1}{2} \int_0^{2\pi} \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} r\, d\theta \;=\; \frac{r^2}{2} \int_0^{2\pi} d\theta \;=\; \pi\, r^2.$$