

The Line Integral

Definition and Properties Let \mathcal{C} be a curve in \mathbf{R}^n , lying in a region \mathcal{R} where a continuous vector field $F(X)$ is also defined. The *line integral*, or *path integral*, of $F(X)$ over the curve \mathcal{C} is denoted by

$$\int_{\mathcal{C}} F(X) \cdot dX, \text{ or } \int_{\mathcal{C}} F(X)^* dX.$$

This will, of course, require some explanation. Let us suppose the endpoints of \mathcal{C} are A and B . We indicate an *orientation* of the curve \mathcal{C} by saying that it proceeds from A to B and we incorporate this into the definition of \mathcal{C} . The same geometric curve, but oriented in the opposite direction from B to A will then be denoted by $-\mathcal{C}$.

We construct a *partition* of the curve \mathcal{C} , consisting of a sequence of points X_k , $k = 0, 1, 2, \dots, N$ on \mathcal{C} with $X_0 = A$, $X_n = B$ and otherwise ordered in the obvious manner; each X_k lies strictly between X_{k-1} and X_{k+1} on the curve. For $k = 1, 2, \dots, N$ we let Ξ_k also be a point on \mathcal{C} , lying between (and not necessarily distinct from) X_{k-1} and X_k . We then form the (scalar) sum

$$\sum_{k=1}^N F(\Xi_k)^* (X_k - X_{k-1}) =$$

$$F(\Xi_1)^* (X_1 - X_0) + F(\Xi_2)^* (X_2 - X_1) + \dots + F(\Xi_N)^* (X_N - X_{N-1}).$$

If we assume that $F(X)$ is continuous and \mathcal{C} is *rectifiable*, which means that there is a positive constant M such that for every partition of \mathcal{C} as described above (in particular, for any number N of points X_k) we have

$$\sum_{k=1}^N \|X_k - X_{k-1}\| \leq M,$$

then one can show the following. Let

$$\delta (= \delta(X_0, X_1, \dots, X_N)) = \max_{k=1,2,\dots,N} \{\|X_k - X_{k-1}\|\}.$$

Then there is a number \mathcal{I} such that

$$\lim_{\delta \rightarrow 0} \sum_{k=1}^N F(\Xi_k)^* (X_k - X_{k-1}) = \mathcal{I}$$

(no matter what collection of partitions we use in letting $\delta \rightarrow 0$, which, at the same time, of course, requires that we let $N \rightarrow \infty$). That number is what we define to be the line integral $\int_{\mathcal{C}} F(X)^* dX$. If

$$F(X) = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{pmatrix},$$

the line integral can also be written in the form

$$\sum_{k=1}^n \int_{\mathcal{C}} f_k(x_1, x_2, \dots, x_n) dx_k = \sum_{k=1}^n \int_{a_k}^{b_k} f_k(x_1, x_2, \dots, x_n) dx_k,$$

where $A = (a_1, a_2, \dots, a_n)^*$, $B = (b_1, b_2, \dots, b_n)^*$. The latter integrals are ordinary (scalar) integrals but to compute them requires some preparation. In the k -th integral it is necessary to express each of x_j , $j = 1, \dots, k-1, k+1, \dots, n$ as a function of x_k . If it is not possible to express one of these, say x_ℓ , as a single valued function of x_k it may be necessary to re-express the integral as a sum of integrals, each involving a separate branch of the multivalued relationship between x_ℓ and x_k .

Example 1 Let \mathcal{C} be the curve in \mathbf{R}^2 which is the graph of $y = x^2$, $0 \leq x \leq 2$, and let

$$F(X) = F(x, y) = \begin{pmatrix} 2xy \\ x^2 + y^2 \end{pmatrix}.$$

Then

$$\int_{\mathcal{C}} F(X)^* dX = \int_{\mathcal{C}} 2xy dx + \int_{\mathcal{C}} (x^2 + y^2) dy.$$

On \mathcal{C} the variable x ranges between 0 and 2 while y ranges between 0 and 4. For the range of x given we have $x = \sqrt{y}$. Thus

$$\int_{\mathcal{C}} F(X)^* dX = \int_0^2 2xy dx + \int_0^4 (x^2 + y^2) dy$$

$$= \int_0^2 2x^3 dx + \int_0^4 (y + y^2) dy = \frac{x^4}{2} \Big|_0^2 + \left(\frac{y^2}{2} + \frac{y^3}{3} \right) \Big|_0^4 = \frac{112}{3}.$$

Example 2 Same field as in Example 1 and the same relationship between x and y but now over the range $-1 \leq x \leq 2$. On $-1 \leq x < 0$ we have $x = -\sqrt{y}$ while on $0 \leq x \leq 2$ we have $x = \sqrt{y}$. Thus $\int_{\mathcal{C}} F(X)^* dX =$

$$\begin{aligned} & \left(\int_{-1}^0 2xy dx + \int_1^0 (x^2 + y^2) dy \right) + \left(\int_0^2 2xy dx + \int_0^4 (x^2 + y^2) dy \right) \\ &= \left(\int_{-1}^0 2x^3 dx + \int_1^0 (y + y^2) dy \right) + \left(\int_0^2 2x^3 dx + \int_0^4 (y + y^2) dy \right) \\ &= \frac{x^4}{2} \Big|_{-1}^0 + \left(\frac{y^2}{2} + \frac{y^3}{3} \right) \Big|_1^0 + \frac{x^4}{2} \Big|_0^2 + \left(\frac{y^2}{2} + \frac{y^3}{3} \right) \Big|_0^4 = 36. \end{aligned}$$

The line integral has many of the standard properties of ordinary integration. In particular, for two fields $F(X)$, $G(X)$ and two scalars α , β we have the *linearity property*

$$\int_{\mathcal{C}} (\alpha F(x) + \beta G(X))^* dX = \alpha \int_{\mathcal{C}} F(X)^* dX + \beta \int_{\mathcal{C}} G(X)^* dX.$$

A further property (note our earlier remarks on orientation) is that

$$\int_{-\mathcal{C}} F(X)^* dX = - \int_{\mathcal{C}} F(X)^* dX.$$

If \mathcal{C}_1 and \mathcal{C}_2 are two curves, the first joining A to B , the second joining B to a third point C , then the sum of these curves, $\mathcal{C}_1 + \mathcal{C}_2$, is the curve obtained by following \mathcal{C}_1 from A to B and then \mathcal{C}_2 from B to C . We then have

$$\int_{\mathcal{C}_1 + \mathcal{C}_2} F(X)^* dX = \int_{\mathcal{C}_1} F(X)^* dX + \int_{\mathcal{C}_2} F(X)^* dX.$$

If the curve \mathcal{C} is represented parametrically via $X(t)$, $a \leq t \leq b$, the line integral can also be expressed and computed in terms of that

parameter, often more easily than the expression in terms of the original variables, x_k , as discussed above. To see this we suppose the points X_k in the partition of the curve \mathcal{C} introduced earlier can be represented as $X_k = X(t_k)$, $a = t_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = b$ while the points Ξ_k can be represented as $\Xi_k = X(\tau_k)$, $t_{k-1} \leq \tau_k \leq t_k$, $k = 1, 2, \dots, N$. Then

$$\begin{aligned} & \sum_{k=1}^N F(\Xi_k)^* (X_k - X_{k-1}) = \\ & F(\Xi_1)^* (X_1 - X_0) + F(\Xi_2)^* (X_2 - X_1) + \dots + F(\Xi_N)^* (X_N - X_{N-1}). \\ & = F(X(\tau_1))^* (X(t_1) - X(t_0)) + F(X(\tau_2))^* (X(t_2) - X(t_1)) + \\ & \quad \dots + F(X(\tau_N))^* (X(t_N) - X(t_{N-1})). \end{aligned}$$

For each k we have

$$X(t_k) - X(t_{k-1}) = \int_{t_{k-1}}^{t_k} X'(s) ds = \left(\frac{\int_{t_{k-1}}^{t_k} X'(s) ds}{t_k - t_{k-1}} \right) (t_k - t_{k-1}).$$

If we assume $X(t)$ is continuously differentiable, then, as $t_k - t_{k-1} \rightarrow 0$,

$$\frac{\int_{t_{k-1}}^{t_k} X'(s) ds}{t_k - t_{k-1}} - X'(\tau_k) \rightarrow 0$$

and we have

$$\begin{aligned} & \sum_{k=1}^N F(\Xi_k)^* (X_k - X_{k-1}) = \\ & = F(X(\tau_1))^* X'(\tau_1) (t_1 - t_0) + F(X(\tau_2))^* X'(\tau_2) (t_2 - t_1) + \\ & \quad \dots + F(X(\tau_N))^* X'(\tau_N) (t_N - t_{N-1}), \end{aligned}$$

which is a Riemann sum tending to the integral

$$\int_a^b F(X(t))^* X'(t) dt$$

as $\delta t = \max_{k=1,2,\dots,N} \{t_k - t_{k-1}\} \rightarrow 0$. In the end we conclude that

$$\int_{\mathcal{C}} F(X)^* dX = \int_a^b F(X(t))^* X'(t) dt.$$

Example 3 If we take $x = t$, $y = t^2$ then the integrals computed in Examples 1 and 2 both take the form

$$\begin{aligned} \int_{\mathcal{C}} (2xy \, dx + (x^2 + y^2) \, dy) &= \int_a^b (2t^3)x'(t) + (t^2 + t^4)y'(t) \, dt \\ &= \int_a^b (2t^3 + (t^2 + t^4)2t) \, dt = \int_a^b (4t^3 + 2t^5) \, dt = \left(t^4 + \frac{t^6}{3} \right) \Big|_a^b. \end{aligned}$$

Taking $a = 0$, $b = 2$ we obtain the result $\frac{112}{3}$, taking $a = -1$, $b = 2$ we obtain the result 36.

One of the reasons for the importance of the line integral lies in its relationship to the physical/mechanical concept of *work*. Work is often defined, in elementary settings, as the product of force times distance. In a multidimensional setting, if a physical object undergoes a displacement, corresponding to a vector dX and is, at the same time, subject to a vector force F , the work done on the object by the force F is the inner product:

$$\text{Work} = F^* dX.$$

If the object moves along a smooth curve \mathcal{C} and is subject to a (possibly) varying force $F(x)$, the analogous statement is that the work done on the object by the force field $F(X)$ is given by

$$\text{Work} = \int_{\mathcal{C}} F(X)^* dX.$$

Many natural force fields are *conservative*; they are formed as the negative gradient of a potential (energy) function $\phi(X)$; $F(X)^* = \nabla\phi(X)$.

Proposition 1 If $F(X)^* = \nabla\phi(X)$, $X \in \mathcal{R} \subset \mathbf{R}^n$, for some continuously differentiable potential function $\phi(X)$ defined in \mathcal{R} , and if \mathcal{C} is

a continuously differentiable (or piecewise continuously differentiable) curve in \mathcal{R} with endpoints A and B (oriented from A to B), then

$$\int_{\mathcal{C}} F(X)^* dX = \int_{\mathcal{C}} \nabla\phi(X) dX = \phi(B) - \phi(A).$$

Remark In other words, the line integral of a gradient field over a curve \mathcal{C} depends only on the endpoints of the curve \mathcal{C} .

Proof Suppose the curve \mathcal{C} is parametrized as

$$\mathcal{C} : X = X(t), \quad a \leq t \leq b, \quad X(a) = A, \quad X(b) = B,$$

with $X(t)$ a continuously differentiable, or piecewise continuously differentiable (this allows \mathcal{C} to have “corners”), function of t . Then we have

$$\int_{\mathcal{C}} \nabla\phi(X) dX = \int_{\mathcal{C}} \nabla\phi(X(t)) \frac{dX}{dt}(t) dt = \int_a^b \nabla\phi(X(t)) X'(t) dt$$

Since the **chain rule** gives $\frac{d}{dt}\phi(X(t)) = \nabla\phi(X(t))X'(t)$, the above yields

$$\int_{\mathcal{C}} \nabla\phi(X) dX = \int_a^b \frac{d}{dt}\phi(X(t)) dt = \phi(X(b)) - \phi(X(a)) = \phi(B) - \phi(A)$$

and we have the result.

Corollary 1 Under the same assumptions as set forth in the preceding proposition, if \mathcal{C}_1 and \mathcal{C}_2 are two curves in \mathcal{R} joining the same two endpoints A and B , both oriented in that direction, then

$$\int_{\mathcal{C}_1} \nabla\phi(X) dX = \int_{\mathcal{C}_2} \nabla\phi(X) dX.$$

Proof Both integrals are equal to $\phi(B) - \phi(A)$.

Implicit in the above proposition and its corollary is the assumption that $\phi(X)$ is a *single-valued* function in \mathcal{R} . The following example shows this to be necessary.

Example 4 Let $\phi(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ in the two dimensional region

$$\mathcal{R} = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 > 0\}.$$

Let \mathcal{C} be the closed curve in \mathcal{R} consisting of the circle of radius R , centered at the origin. Here we have, in \mathcal{R} ,

$$\frac{\partial \phi}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2}.$$

Parametrizing the curve \mathcal{C} via $x = x(\theta) = R \cos \theta$, $y = y(\theta) = R \sin \theta$, $0 \leq \theta \leq 2\pi$, we observe that, on \mathcal{C} , we have $x^2 + y^2 = R^2 \cos^2 \theta + R^2 \sin^2 \theta = R^2$ and thus

$$\begin{aligned} \int_{\mathcal{C}} \nabla \phi(X) dX &= \int_{\mathcal{C}} \left(\frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \right) \\ &= \int_0^{2\pi} \left(\frac{-y(\theta)}{x(\theta)^2 + y(\theta)^2} x'(\theta) + \frac{x(\theta)}{x(\theta)^2 + y(\theta)^2} y'(\theta) \right) d\theta \\ &= \int_0^{2\pi} \left(\frac{-R \sin \theta}{R^2} (-R \sin \theta) + \frac{R \cos \theta}{R^2} (R \cos \theta) \right) d\theta \\ &= \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) d\theta = \int_0^{2\pi} 1 d\theta = 2\pi, \end{aligned}$$

which is certainly not zero. The reason why this does not contradict the results set forth earlier is that $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, being the angular coordinate in the plane, is not single valued; if a point with Cartesian coordinates (x, y) corresponds to an angle θ , then it also corresponds to the angle $\theta + 2k\pi$ for any integer k . As we go around the circle of radius R we start with $\theta = \tan^{-1}\left(\frac{y}{x}\right) = 0$ at the point $(R, 0)$ and end up at that point again, but with $\theta = \tan^{-1}\left(\frac{y}{x}\right) = 2\pi$ at the end.