The Line Integral

Definition and Properties

Let $C$ be a curve in $\mathbb{R}^n$, lying in a region $\mathcal{R}$ where a continuous vector field $F(X)$ is also defined. The line integral, or path integral, of $F(X)$ over the curve $C$ is denoted by

$$\int_C F(X) \cdot dX, \text{ or } \int_C F(X)^*dX.$$ 

This will, of course, require some explanation. Let us suppose the endpoints of $C$ are $A$ and $B$. We indicate an orientation of the curve $C$ by saying that it proceeds from $A$ to $B$ and we incorporate this into the definition of $C$. The same geometric curve, but oriented in the opposite direction from $B$ to $A$ will then be denoted by $-C$.

We construct a partition of the curve $C$, consisting of a sequence of points $X_k$, $k = 0, 1, 2, ..., N$ on $C$ with $X_0 = A$, $X_N = B$ and otherwise ordered in the obvious manner; each $X_k$ lies strictly between $X_k-1$ and $X_{k+1}$ on the curve. For $k = 1, 2, ..., N$ we let $\Xi_k$ also be a point on $C$, lying between (and not necessarily distinct from) $X_{k-1}$ and $X_k$. We then form the (scalar) sum

$$\sum_{k=1}^{N} F(\Xi_k)^* (X_k - X_{k-1}) = F(\Xi_1)^* (X_1 - X_0) + F(\Xi_2)^* (X_2 - X_1) + \cdots + F(\Xi_N)^* (X_N - X_{N-1}).$$

If we assume that $F(X)$ is continuous and $C$ is rectifiable, which means that there is a positive constant $M$ such that for every partition of $C$ as described above (in particular, for any number $N$ of points $X_k$) we have

$$\sum_{k=1}^{N} \|X_k - X_{k-1}\| \leq M,$$

then one can show the following. Let

$$\delta ( = \delta (X_0, X_1, ..., X_N)) = \max_{k=1,2,\ldots,N} \{\|X_k - X_{k-1}\|\}.$$
Then there is a number $I$ such that
\[
\lim_{\delta \to 0} \sum_{k=1}^{N} F(\Xi_k)^* (X_k - X_{k-1}) = I
\]
(no matter what collection of partitions we use in letting $\delta \to 0$, which, at the same time, of course, requires that we let $N \to \infty$). That number is what we define to be the line integral $\int_C F(X)^* dX$. If
\[
F(X) = \left( f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n) \right),
\]
the line integral can also be written in the form
\[
\sum_{k=1}^{n} \int_C f_k(x_1, x_2, \ldots, x_n) \, dx_k = \sum_{k=1}^{n} \int_{a_k}^{b_k} f_k(x_1, x_2, \ldots, x_n) \, dx_k,
\]
where $A = (a_1, a_2, \ldots, a_n)^*$, $B = (b_1, b_2, \ldots, b_n)^*$. The latter integrals are ordinary (scalar) integrals but to compute them requires some preparation. In the $k$-th integral it is necessary to express each of $x_j$, $j = 1, \ldots, k-1, k+1, \ldots, n$ as a function of $x_k$. If it is not possible to express one of these, say $x_\ell$, as a single valued function of $x_k$ it may be necessary to re-express the integral as a sum of integrals, each involving a separate branch of the multivalued relationship between $x_\ell$ and $x_k$.

**Example 1** Let $C$ be the curve in $\mathbb{R}^2$ which is the graph of $y = x^2$, $0 \leq x \leq 2$, and let
\[
F(X) = F(x, y) = \left( \frac{2xy}{x^2 + y^2} \right).
\]
Then
\[
\int_C F(X)^* dX = \int_C 2xy \, dx + \int_C (x^2 + y^2) \, dy.
\]
On $C$ the variable $x$ ranges between 0 and 2 while $y$ ranges between 0 and 4. For the range of $x$ given we have $x = \sqrt{y}$. Thus
\[
\int_C F(X)^* dX = \int_0^2 2xy \, dx + \int_0^4 (x^2 + y^2) \, dy
\]
\[ = \int_{0}^{2} 2x^3 \, dx + \int_{0}^{4} (y + y^2) \, dy = \frac{x^4}{2}\bigg|_{0}^{2} + \left(\frac{y^2}{2} + \frac{y^3}{3}\right)\bigg|_{0}^{4} = \frac{112}{3}. \]

**Example 2**  
Same field as in Example 1 and the same relationship between \( x \) and \( y \) but now over the range \(-1 \leq x \leq 2\). On \(-1 \leq x < 0\) we have \( x = -\sqrt{y} \) while on \( 0 \leq x \leq 2\) we have \( x = \sqrt{y} \). Thus

\[
\int_{C} F(X)^* dX = \left( \int_{-1}^{0} 2xy \, dx + \int_{1}^{0} (x^2 + y^2) \, dy \right) + \left( \int_{0}^{2} 2xy \, dx + \int_{0}^{4} (x^2 + y^2) \, dy \right) = \left( \int_{-1}^{0} 2x^3 \, dx + \int_{1}^{0} (y + y^2) \, dy \right) + \left( \int_{0}^{2} 2x^3 \, dx + \int_{0}^{4} (y + y^2) \, dy \right) = \frac{x^4}{2}\bigg|_{-1}^{0} + \left(\frac{y^2}{2} + \frac{y^3}{3}\right)\bigg|_{1}^{0} + \frac{x^4}{2}\bigg|_{0}^{2} + \left(\frac{y^2}{2} + \frac{y^3}{3}\right)\bigg|_{0}^{4} = 36.
\]

The line integral has many of the standard properties of ordinary integration. In particular, for two fields \( F(X) \), \( G(X) \) and two scalars \( \alpha, \beta \) we have the linearity property

\[
\int_{C} (\alpha F(x) + \beta G(X))^* dX = \alpha \int_{C} F(X)^* dX + \beta \int_{C} G(X)^* dX.
\]

A further property (note our earlier remarks on orientation) is that

\[
\int_{-C} F(X)^* dX = -\int_{C} F(X)^* dX.
\]

If \( C_1 \) and \( C_2 \) are two curves, the first joining \( A \) to \( B \), the second joining \( B \) to a third point \( C \), then the sum of these curves, \( C_1 + C_2 \), is the curve obtained by following \( C_1 \) from \( A \) to \( B \) and then \( C_2 \) from \( B \) to \( C \). We then have

\[
\int_{C_1+C_2} F(X)^* dX = \int_{C_1} F(X)^* dX + \int_{C_2} F(X)^* dX.
\]

If the curve \( C \) is represented parametrically via \( X(t), \ a \leq t \leq b \), the line integral can also be expressed and computed in terms of that
parameter, often more easily than the expression in terms of the original variables, $x_k$, as discussed above. To see this we suppose the points $X_k$ in the partition of the curve $C$ introduced earlier can be represented as $X_k = X(t_k)$, $a = t_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b$ while the points $\Xi_k$ can be represented as $\Xi_k = X(\tau_k)$, $t_{k-1} \leq \tau_k \leq t_k$, $k = 1, 2, \ldots, N$. Then

$$\sum_{k=1}^{N} F(\Xi_k)^* (X_k - X_{k-1}) =$$

$$= F(\Xi_1)^* (X_1 - X_0) + F(\Xi_2)^* (X_2 - X_1) + \cdots + F(\Xi_N)^* (X_N - X_{N-1}).$$

$$= F(X(\tau_1))^* (X(t_1) - X(t_0)) + F(X(\tau_2))^* (X(t_2) - X(t_1)) +$$

$$\cdots + F(X(\tau_N))^* (X(t_N) - X(t_{N-1})).$$

For each $k$ we have

$$X(t_k) - X(t_{k-1}) = \int_{t_{k-1}}^{t_k} X'(s) \, ds = \left( \frac{\int_{t_{k-1}}^{t_k} X'(s) \, ds}{t_k - t_{k-1}} \right) (t_k - t_{k-1}).$$

If we assume $X(t)$ is continuously differentiable, then, as $t_k - t_{k-1} \to 0$,

$$\frac{\int_{t_{k-1}}^{t_k} X'(s) \, ds}{t_k - t_{k-1}} - X'(\tau_k) \to 0$$

and we have

$$\sum_{k=1}^{N} F(\Xi_k)^* (X_k - X_{k-1}) =$$

$$= F(X(\tau_1))^* X'(\tau_1) (t_1 - t_0) + F(X(\tau_2))^* X'(\tau_2) (t_2 - t_1) +$$

$$\cdots + F(X(\tau_N))^* X'(\tau_N) (t_N - t_{N-1}),$$

which is a Riemann sum tending to the integral

$$\int_{a}^{b} F(X(t))^* X'(t) \, dt$$

as $\delta t = \max_{k=1,2,\ldots,N} \{t_k - t_{k-1}\} \to 0$. In the end we conclude that

$$\int_{C} F(X)^* \, dX = \int_{a}^{b} F(X(t))^* X'(t) \, dt.$$
Example 3  If we take \( x = t, \ y = t^2 \) then the integrals computed in Examples 1 and 2 both take the form

\[
\int_{C} (2xy \, dx + (x^2 + y^2) \, dy) = \int_{a}^{b} \left( (2t^3) x'(t) + (t^2 + t^4) y'(t) \right) \, dt
\]

\[
= \int_{a}^{b} \left( 2t^3 + (t^2 + t^4) \right) \, dt = \left[ t^4 + \frac{t^6}{3} \right]_{a}^{b}.
\]

Taking \( a = 0, \ b = 2 \) we obtain the result \( \frac{112}{3} \), taking \( a = -1, \ b = 2 \) we obtain the result 36.

One of the reasons for the importance of the line integral lies in its relationship to the physical/mechanical concept of work. Work is often defined, in elementary settings, as the product of force times distance. In a multidimensional setting, if a physical object undergoes a displacement, corresponding to a vector \( dX \) and is, at the same time, subject to to a vector force \( F \), the work done on the object by the force \( F \) is the inner product:

\[
\text{Work} = F^* \, dX.
\]

If the object moves along a smooth curve \( C \) and is subject to a (possibly) varying force \( F(x) \), the analogous statement is that the work done on the object by the force field \( F(X) \) is given by

\[
\text{Work} = \int_{C} F(X)^* \, dX.
\]

Many natural force fields are conservative; they are formed as the negative gradient of a potential (energy) function \( \phi(X) \); \( F(X)^* = -\nabla \phi(X) \).

Proposition 1 If \( F(X)^* = -\nabla \phi(X), \ X \in \mathcal{R} \subset \mathbb{R}^n \), for some continuously differentiable potential function \( \phi(X) \) defined in \( \mathcal{R} \), and if \( C \) is
a continuously differentiable (or piecewise continuously differentiable) curve in \( \mathcal{R} \) with endpoints \( A \) and \( B \) (oriented from \( A \) to \( B \)), then

\[
\int_{\mathcal{C}} F(X)^*dX = \int_{\mathcal{C}} \nabla \phi(X) dX = \phi(B) - \phi(A).
\]

**Remark** In other words, the line integral of a gradient field over a curve \( \mathcal{C} \) depends only on the endpoints of the curve \( \mathcal{C} \).

**Proof** Suppose the curve \( \mathcal{C} \) is parametrized as

\[
\mathcal{C}: X = X(t), \ a \leq t \leq b, \ X(a) = A, \ X(b) = B,
\]

with \( X(t) \) a continuously differentiable, or piecewise continuously differentiable (this allows \( \mathcal{C} \) to have “corners”), function of \( t \). Then we have

\[
\int_{\mathcal{C}} \nabla \phi(X) dX = \int_{\mathcal{C}} \nabla \phi(X(t)) \frac{dX(t)}{dt} dt = \int_a^b \nabla \phi(X(t)) X'(t) dt
\]

Since the **chain rule** gives \( \frac{d}{dt} \phi(X(t)) = \nabla \phi(X(t)) X'(t) \), the above yields

\[
\int_{\mathcal{C}} \nabla \phi(X) dX = \int_a^b \frac{d}{dt} \phi(X(t)) dt = \phi(X(b)) - \phi(X(a)) = \phi(B) - \phi(A)
\]

and we have the result.

**Corollary 1** Under the same assumptions as set forth in the preceding proposition, if \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are two curves in \( \mathcal{R} \) joining the same two endpoints \( A \) and \( B \), both oriented in that direction, then

\[
\int_{\mathcal{C}_1} \nabla \phi(X) dX = \int_{\mathcal{C}_2} \nabla \phi(X) dX.
\]

**Proof** Both integrals are equal to \( \phi(B) - \phi(A) \).
Implicit in the above proposition and its corollary is the assumption that \( \phi(X) \) is a *single-valued* function in \( \mathcal{R} \). The following example shows this to be necessary.

**Example 4** Let \( \phi(x, y) = \tan^{-1}\left(\frac{y}{x}\right) \) in the two dimensional region

\[
\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 0\}.
\]

Let \( \mathcal{C} \) be the closed curve in \( \mathcal{R} \) consisting of the circle of radius \( R \), centered at the origin. Here we have, in \( \mathcal{R} \),

\[
\frac{\partial \phi}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2}.
\]

Parametrizing the curve \( \mathcal{C} \) via \( x = x(\theta) = R \cos \theta, \quad y = y(\theta) = R \sin \theta \), \( 0 \leq \theta \leq 2\pi \), we observe that, on \( \mathcal{C} \), we have \( x^2 + y^2 = R^2 \cos^2 \theta + R^2 \sin^2 \theta = R^2 \) and thus

\[
\int_{\mathcal{C}} \nabla \phi(X) \, dX = \int_{\mathcal{C}} \left( -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \right)
\]

\[
= \int_{0}^{2\pi} \left( -\frac{y(\theta)}{x(\theta)^2 + y(\theta)^2} \, x'(\theta) + \frac{x(\theta)}{x(\theta)^2 + y(\theta)^2} \, y'(\theta) \right) \, d\theta
\]

\[
= \int_{0}^{2\pi} \left( \frac{-R \sin \theta}{R^2} (-R \sin \theta) + \frac{R \cos \theta}{R^2} (R \cos \theta) \right) \, d\theta
\]

\[
= \int_{0}^{2\pi} (\sin^2 \theta + \cos^2 \theta) \, d\theta = \int_{0}^{2\pi} 1 \, d\theta = 2\pi,
\]

which is certainly not zero. The reason why this does not contradict the results set forth earlier is that \( \theta = \tan^{-1}\left(\frac{y}{x}\right) \), being the angular coordinate in the plane, is not single valued; if a point with Cartesian coordinates \((x, y)\) corresponds to an angle \( \theta \), then it also corresponds to the angle \( \theta + 2k\pi \) for any integer \( k \). As we go around the circle of radius \( R \) we start with \( \theta = \tan^{-1}\left(\frac{y}{x}\right) = 0 \) at the point \((R, 0)\) and end up at that point again, but with \( \theta = \tan^{-1}\left(\frac{y}{x}\right) = 2\pi \) at the end.